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ABSTRACT

This twenty-fourth unit in the SMSG secondary school mathematics series is the teacher's commentary for Unit 23. For each of the chapters in Unit 23, a time allotment is suggested, the goals for that chapter are discussed, the mathematics is explained, some teaching suggestions are given, and answers to exercises are provided. In the appendix is a general discussion of the research exercises described in the appendix of the student's text, followed by the mathematical details for each of the four research exercises. (DT)

School Mathematics Study Group

Introduction to Matrix Algebra

Unit 24

Introduction to Matrix Algebra

Teacher's Commentary

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PREFACE

The text that the present "Commentary for Teachers" accompanies represents a brave attempt to further the introduction of modern mathematics into the secondary-school curriculum. Except for isolated experiments, the subject of matrices has not heretofore been taught at the high-school level. The results of these few isolated experiments, however, have been so rewarding and so successful that all teachers should have courage regarding the exciting possibilities in this material. Here is some truly modern mathematics that is eminently useful and that can be understood by all college-capable boys and girls.

The text has been arranged so that individual chapters make separate units. For the class that has little time for the subject, Chapter 1, which treats the operations of multiplication and addition, makes a unit. There is much to be gained even from such a small unit, since in it the students will be introduced to a meaningful example of noncommutative multiplication. It is assumed that the students have previously heard about the commutative, associate, and distributive laws. Certainly an understanding of these laws should be a part of their early training in algebra. But since these students have had little or no experience with number systems other than those of the real and complex numbers, they will perhaps not completely comprehend the full significance of the laws. To demonstrate to the students a new number system in which the commutative law does not hold is most worthwhile. Since the ideas of Chapter 1 are simple and there is a great deal of manipulation, largely arithmetical, the chapter will serve as an easy introduction to the more difficult ideas contained in subsequent chapters.

The next three chapters are quite independent of each other. Chapter 2 is the most important from the mathematician's point of view. In this chapter, a subset of matrices, the set of 2×2 matrices, is considered in detail. Most pupils who study secondary school mathematics complete their study believing that there is just one "algebra." Indeed, they do not know quite what "an algebra" is. Through the study of the very neat algebraic system associated with this subset of the 2×2 matrices, the concept of an algebra will be understood much more clearly. The meaning of the important mathematical notion of an inverse will also be more thoroughly comprehended. Best of all, the logical aspects of the chapter are developed carefully and rigorously. It is assumed that all students entering the course will already have a considerable knowledge of axiomatic systems, gained through the study of geometry. In this chapter, axiomatic methods are applied to algebraic systems. There are many proofs, and no statements are made unless supported by rigorous demonstration. Chapter 2 undoubtedly contains more "mathematics" than Chapter 1.

In many algebra books used in courses commonly called "advanced algebra," reference is made to the use of determinants in the solution of linear equations. Usually the subject is presented without mentioning matrices. In Chapter 3, it is clearly seen that determinants are a small portion of a much more extensive subject. The study of matrices adds greatly to our understanding and facility in solving systems of linear equations and leads naturally toward more advanced considerations in collegiate mathematics.

A shift in point of view is made in Chapter 4. In the study of sciences, particularly physics, many students are already familiar with the idea of a vector. In Chapter 4, a vector is introduced as an array of numbers. The algebra of vectors is developed together with the geometric interpretation. Chapter 4 is not dependent on either Chapter 2 or Chapter 3.

Chapter 5 should be studied only after Chapter 4 has been covered. It advances rapidly in the study of transformations of the plane. This beautiful basic application of matrix theory ties together much that the student has learned concerning algebra, geometry, trigonometry, and functions, and thus it furnishes a fitting capstone to his secondary-school study of mathematics.

As an added teaser, however, a delightful set of "research exercises" has been appended to point toward more exciting mathematics ahead!

The entire book can be studied in a half-year course for college-capable students. This means that a large amount of extremely significant mathematics will be met in a short space of time.

The text is flexible and can be adapted to various types of classes. For a minimum course of one month, Chapter 1 can be studied. A longer course with a class of able pupils could consist of Chapter 1 together with Chapter 2, or Chapter 3, or Chapter 4. Indeed, Chapter 1 together with any combination of Chapters 2, 3, and 4 constitutes a unit. As indicated above, Chapter 5 should be studied only in combination with Chapter 4. The four research exercises of the Appendix are considerably dependent, for their full appreciation, on the material in Chapter 2; they are quite independent of Chapters 3, 4, and 5.

A suggested time schedule is the following:

Chapter 1 — 2 weeks
Chapter 2 — 4 weeks

Chapter 3	—	2 weeks
Chapter 4	—	3 weeks
Chapter 5	—	3 weeks
Appendix	—	4 weeks

A considerable amount of collateral reading is recommended. This reading has the purpose of broadening the students' understanding of the nature of mathematics. It is assumed that the class will already be familiar with many of the notions of sets; if not, the first assignment of collateral reading should be in this area. Here are the titles of some books that, along with those listed in the Bibliography on page 231 of the accompanying text, will be found useful:

I. Adler, "The New Mathematics," John Day Company, New York, 1958.

E. T. Bell, "Mathematics, Queen and Servant of Science," McGraw-Hill Book Company, Inc., New York, 1951.

George A. W. Boehm, "The New World of Math," The Dial Press, New York, 1959.

W. W. Sawyer, "Mathematicians Delight," Penguin Books, Inc., Baltimore, 1957.

Chapter 1
MATRIX OPERATIONS

1-1. Introduction

In the Introduction to Chapter 1, the text moves very slowly. If necessary, you can handle all the material in this first section in one class period. This is not a wise thing to do, however, and should be avoided if possible. In their experience with mathematics, the students have become much more rigid than the teacher might like to concede. One of the primary purposes of the book is to give the students some awareness of the breadth and scope of mathematics. In order to prepare them for the work to come, it is well to spend several days on the Introduction. If the pupils have already had some experience in developing a number system, so much the better. If they have not had this experience, it would be wise to study the system of rational numbers a/b in terms of ordered pairs (a,b) of integers, with b a counting number. There are two great advantages to the ordered-pair concept. The first is the traditional value: The number system is extended logically as the postulates become less restrictive. The second value is the development of the concept of an ordered pair being a single entity, in preparation for handling the more advanced concept of an entire matrix as an entity.

It is within the capacity of most students who study rigorous mathematics in the twelfth grade to invent a number system. Once the pupil understands the relationships between definitions, postulates, and theorems, he can devise his own number system. To be significant, however, the number system should satisfy two very important criteria. The first is that the postulates prove fruitful, that from the set of postulates alone many theorems can be developed. The second is that the mathematical system, when developed, prove useful in having interesting applications. If the study of mathematics can be made an adventure, the students will be eager in their learning.

A rectangular array of numbers is called a matrix. In this text, we shall enclose each matrix in a pair of square brackets $[\quad]$. There is no universal agreement for this convention. Some authors use $(\ )$, and others use $\{\ \}$. Note that a single number, such as 3, enclosed in square brackets, denotes a matrix. As the student develops mathematical sophistication, he will understand that the notions inherent in the symbol 3 and those inherent in the symbol $\begin{bmatrix} 3 \end{bmatrix}$ are different.

Historically, as noted by C. C. MacDuffee in "What Is a Matrix?", American Mathematical Monthly, vol. 50 (1943), pp. 360–365, the term matrix was introduced into mathematics in 1850 by J. J. Sylvester: "We commence with an oblong arrangement of terms consisting of m lines and n columns. This will not in itself represent a determinant, but is as it were, a Matrix out of which we may form systems of determinants by fixing upon a number p, and selecting p lines and p columns, the squares corresponding to which may be termed determinants of the pth order."

W. R. Hamilton used matrix algebra in [illegible] linear and vector functions. In 1855, Arthur Cayley referr[illegible] "as being very convenient notation for the theory of linear equations," and added the casual comment that "there are surely many things to be said about this theory of matrices." In 1858, he returned to the systematic development of their properties, as here presented in Chapter 1.

1-2. The Order of a Matrix

In this text, we shall speak of the "order" of a matrix. Another frequently used term is "dimension." The word "dimension" in many ways is a more natural term, since we are speaking of two quantities — the number of rows and the number of columns. However, it is well to reserve the word "dimension" for less technical discussions. The word "order" will be given a unique mathematical meaning that will facilitate better communication between instructor and student once the idea is understood. Thereafter, the student can use "dimension" without being involved in technical uses of the word.

In referring to a square matrix, it is not necessary to designate two numbers. For instance, in referring to a 2×2 square matrix, it is sufficient to speak of a square matrix of order 2.

Little attention need be paid in this chapter to the concept of a row matrix or a row vector (see "The Mathematics Teacher," January, 1960). The subject of vectors is explored at length in Chapters 4 and 5. At this time, it is sufficient merely to introduce the term. It is important, however, to differentiate between a point having coordinates such as $(2, 3)$ and a row vector $\begin{bmatrix} 2 & 3 \end{bmatrix}$. Although there is a geometrical representation of row vectors that involves points, there are two distinct concepts to be considered. Both concepts are valuable, and an effort should be made to understand the difference between them. It is worth noting at this point that a very interesting short course can be given that would involve Chapter 1, Chapter 4, and possibly Chapter 5.

If the class has not had previous experience with subscript notation, a considerable number of exercises should be devoted to drill in this terminology since it will be encountered frequently throughout the book. The two letters i and j can be considered as variables of which the range must be designated. The usual range for i is $\{1,2,\ldots,m\}$, which means that i takes on each value between 1 and m, inclusive. The usual range for j is $\{1,2,\ldots,n\}$. Thus, if m equals 4 and n equals 6, the notation a_{ij} is a general representation for <u>each one</u> of twenty-four entries. Note that it is important to think of a_{ij} as representing each entry separately, not all entries together. Attention should be focused on one entry at a time, and in this connection there should not be consideration of all entries at one time in a kind of amorphous mass.

There are three rather common notations for the transpose of a matrix A. These are A^T, A^t, and A'. Although the last notation may be the most common, it has not been used in this book since the prime notation does not impress the consciousness as much as the others. For students in secondary school, it is safer to use A^T or A^t. Many theorems involving the transpose are developed later, in Chapter 2; they have been introduced here for the simple reason that they afford convenient material for practice in dealing with matrices and their elements.

To help the class further to familiarize itself with rows, columns, entries, etc., you might have the members engage in a little game involving such a matrix as

$$\begin{array}{cc} & \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{array}{c} R_1 \\ R_2 \end{array} & \begin{bmatrix} 1 & -3 & 2 \\ -1 & 3 & -2 \end{bmatrix} \end{array}.$$

Positive entries represent gains for the first player and losses for the second, while negative entries represent gains for the second player and losses for the first.

To play the game, the first player writes the number of a row on his paper, and the second player writes the number of a column. When the numbers are announced, the entry at the intersection of the chosen row and column is marked down as the score for that play of the game. Thus if the choices are R_2 and C_1, then the score is -1. At the end of 10 plays, the scores are added and the first player wins or loses according as the sum is positive or negative. Simple as such matrix games are, they are representative of the competitive situations that exist, for example, in business and war. In the late 1920's, they led the great modern mathematician John von Neumann (1903–1957) to the founding of a new branch of mathematics, the Theory of Games; see the delightful book by J. D. Williams, The Complete Strategyst, McGraw-Hill Book Company, Inc., New York, 1954. This theory has had a great impact on economics and other sciences.

1–2

1. (a) The students will likely bring in examples from newspapers, magazines, and books. These might involve the stock market, health statistics, mileages, agricultural production, armaments, populations, etc.

   (b) The order $m \times n$ is the number m of horizontal rows, followed by the number n of vertical columns. For example, the matrix

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

   is of order 3×2.

   (c) Alternative methods involve sentences, graphs, etc.

2. (a) For example: $\begin{bmatrix} 17 & 62 & 124 \end{bmatrix}$.

   (b) Such a vector might be used in organizing games, etc. More extensive information of a similar sort is employed, for example, in identifying people by their finger prints.

3. (a) 4×5. (b) 0, 3, −7, 8, 7. (c) 3, 12, −5, −7.

   (d) −7. (e) 4. (f) 0.

   (g) $\begin{bmatrix} 1 & 8 & -1 & 0 \\ 2 & 10 & -3 & 3 \\ 3 & 12 & -5 & -7 \\ 4 & 14 & 6 & 8 \\ 5 & 16 & 3 & 7 \end{bmatrix}$.

4\. (a) 4×4. (b) 0, 0, 1, 0. (c) 0, 0, 1, 0.

(d) 0. (e) For $i = j$. (f) For $i \neq j$.

(g) $B^t = B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

5\. Examples:

(a) $\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$.

(b) $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \pi & e \\ 0.6 & 0.[illegible] & [illegible] & [illegible] \\ \frac{2}{3} & \frac{1}{2} & \sqrt{2} & \sqrt{3} \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 \end{bmatrix}$.

6\. (a) 4. (b) 12. (c) n^2. (d) mn.

1-3. Equality of Matrices

It should be noted that no postulates have been assumed for the "equals" relationship. The equivalence properties for equality (i.e., the reflexive, symmetric, and transitive properties) are inherent in the given definition of equality of matrices. If these properties have been discussed previously, it can be demonstrated that they are satisfied by the definition of equality of matrices; otherwise they probably should not be stressed at this time. It is very likely that postulates involving equality and operations, such as "if equals are added to equals, the sums are equal," will appear in student proofs involving matrices. This point might be discussed when an opportunity arises naturally in a classroom discussion.

Note that $\begin{bmatrix} 0 & 0 \end{bmatrix}$ does not equal $\begin{bmatrix} 0 \end{bmatrix}$. Under our definition, two matrices must have the same order if they are to be equal. Since these two matrices are not of the same order, they cannot be equal.

[pages 6-8]

Exercises 1-3

1. (a) $x = -1, \quad y = -1.$

   (b) $x = 1, \quad y = 2.$

   (c) The four equations,

$$x^2 = 1, \quad y = -1,$$
$$x = -1, \quad y^2 = 1,$$

   are consistent. The unique solution is $x = -1, \quad y = -1.$

2. If matrix A = matrix B, then the matrices are of the same order and their corresponding entries are equal. Thus $a_{ij} = b_{ij}$ for all permissible i, j. If $B = C$, then also $b_{ij} = c_{ij}$. Hence $a_{ij} = c_{ij}$ for all permissible i, j, so that $A = C$ by Definition 1-2.

3. $\begin{bmatrix} 1 & 4 & 7 \\ 3 & 6 & 9 \end{bmatrix}.$

4. $\begin{bmatrix} 1 & 2 \\ -1 & 3 \\ -1 & 5 \\ 0 & 1 \end{bmatrix}.$

5. $\begin{bmatrix} 1 & -2 \\ 5 & -5 \\ -5 & 3 \\ 0 & 1 \end{bmatrix}.$

1-4. Addition of Matrices

In this section, the operation of addition is developed slowly and carefully.

Stress the fact that the definition of addition does not give a rule by which matrices of different orders could be added. Given the problem: Find the sum of the two quantities

$$\begin{bmatrix} \text{[illegible]} & 3 & -1 \\ 4 & 5 & 2 \\ 1 & 0 & 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 0 \\ 3 & -1 \end{bmatrix},$$

some students will be tempted to enlarge the second matrix by adding a row and a column of zeros. It should be stressed that this produces an altogether different matrix. Under our definition of addition, the sum of the two matrices given above is undefined.

The commutative law and the associative law for the addition of matrices should not be belabored at this time. It is quite obvious that the addition of matrices does possess these two properties. When multiplication is considered, then the commutative property for the addition of matrices can be put in sharp contrast with the failure of the commutative property for the multiplication of matrices.

Although many students will be inclined to pass over the three theorems at the end of this section by dismissing them as 'obvious,' the proofs involve a considerable amount of worthwhile algebra. Proofs of these theorems will sharpen the understanding of the relationships between definitions, postulates, and theorems. (See Exercises 11 through 14 in this section.)

Exercises 1-4

1. The single matrix equation is equivalent to the six real-number equations,

$$\begin{aligned} x + 3 &= 0, & 2y - 8 &= -6, \\ a + 1 &= -3, & 4x + 6 &= 2x, \\ b - 3 &= 2b + 4, & 3b &= -21, \end{aligned}$$

or

$$\begin{aligned} x &= -3, & y &= 1, \\ a &= -4, & x &= -3, \\ b &= -7, & b &= -7, \end{aligned}$$

so that the equations are consistent and have a unique solution. Two of the equations are redundancies.

2. (a) $8 + 2 = 10$. (b) $1 + 8 = 9$. (c) $4 + 4 = 8$.

3. $\begin{bmatrix} \frac{1}{3} & \frac{4}{21} \\ \frac{1}{8} & \frac{4}{45} \end{bmatrix}$.

4. $\begin{bmatrix} \frac{3}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{5} & \frac{7}{6} & \frac{1}{7} \\ \frac{1}{8} & \frac{1}{9} & \frac{11}{10} \end{bmatrix}$.

5. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

6. (a) No. You cannot add matrices of different orders.

   (b) Yes.

   (c) Same as first matrix.

7. $\begin{bmatrix} 4 & 2 & -2 \\ -9 & 9 & 3 \\ 8 & -4 & 3 \end{bmatrix}$.

8. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

9. (a) $A + B = \begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 5 & 7 \end{bmatrix}$.

   (b) $(A + B) + C = \begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 5 & 7 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 1 & 0 \\ -2 & -4 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 7 & 2 \\ 3 & 3 \end{bmatrix}$.

   (c) $A + (B + C) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 6 & 1 \\ 4 & -2 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 7 & 2 \\ 3 & 3 \end{bmatrix}$.

   (d) $A - B = \begin{bmatrix} -1 & 3 \\ 0 & 6 \\ 5 & 5 \end{bmatrix}$.

   (e) $(A - B) + C = \begin{bmatrix} -1 & 3 \\ 0 & 6 \\ 5 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 1 & 0 \\ -2 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 6 \\ 3 & 1 \end{bmatrix}$.

   (f) $B - A = \begin{bmatrix} 1 & -3 \\ 0 & -6 \\ -5 & -5 \end{bmatrix}$.

10. (a) The associative law for addition.

    (b) $A - B = -(B - A)$.

11. The entry in the i-th row and j-th column of $-(-A)$ is $-(-a_{ij}) = a_{ij}$ by the laws of real numbers. But this is the entry of A in the i-th

[pages 15-17]

row and j-th column; and if two matrices have equal entries at all corresponding positions, they are equal.

12. Since $-0 = 0$, every entry of $-\underline{0}$ is equal to the corresponding entry of $\underline{0}$, so that the matrices are equal.

13. Since

$$\begin{aligned}-(a_{ij} + b_{ij}) &= -a_{ij} - b_{ij}\\ &= (-a_{ij}) + (-b_{ij}),\end{aligned}$$

the corresponding entries of $-(A + B)$ and $(-A) + (-B)$ are equal, so that the matrices also are equal.

14. To prove that $A^t + B^t = (A + B)^t$, we simply have to show that the entries in the same row and column are equal. Let

$$A + B = C.$$

Then

$$c_{ij} = a_{ij} + b_{ij}.$$

Now

$$a_{ij}^t = a_{ji}, \quad b_{ij}^t = b_{ji}, \quad c_{ij}^t = c_{ji}.$$

But

$$c_{ji} = a_{ji} + b_{ji} = a_{ij}^t + b_{ij}^t.$$

Hence

$$c_{ij}^t = a_{ij}^t + b_{ij}^t,$$

or

$$C^t = A^t + B^t.$$

But

$$C = A + B,$$

so

$$C^t = (A + B)^t,$$

and therefore

$$(A + B)^t = A^t + B^t.$$

1-5. Addition of Matrices (Concluded)

Insofar as only addition and subtraction are involved, the algebra of matrices is exactly like the ordinary algebra of numbers. This statement is underlined in the text. In order to provide a sharp parallel, the introduction to the subject of groups may begin here. The real numbers, under the operation of addition, form a group; that is, they satisfy the postulates of closure, associativity, identity, and inverse. Also this group is an abelian group since the commutative property holds for it. The set of all 2×2 matrices, such as

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \quad \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

forms a group under the operation of addition. Also the set of all 3×3 matrices forms a group under addition. Through the use of the group concept, the structure of the mathematics can be spotlighted. In Chapter 2, the group concept is developed. Mention of the concept and a brief discussion at this time will serve as an introduction to the later formal consideration.

In order to solve the matrix equation

$$X + A = B,$$

we add the additive inverse of A, namely $-A$, to both sides of the equation. Once again students will be tempted to say, "Transpose," or, "Put A on the other side and change signs." Both practices should be avoided, since they diminish understanding. Because the inverse has been emphasized considerably, it is doubtful if the student will depend on these mechanical conveniences. It is important to emphasize and drill the notion of an additive inverse, that is, a matrix that 'neutralizes' the result of addition:

$$X + A + (-A) = X, \quad A + (-A) = \underline{0}.$$

Exercises 1–5

1. $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$

2. $X = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 3 \\ 4 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 3 \\ 3 & 3 & 4 \end{bmatrix}.$

3. $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} -6 & 2 & -3 \end{bmatrix} + \begin{bmatrix} -6 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -12 & 2 & -1 \end{bmatrix}.$

4. $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}.$

5. $-\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 5 & -1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 4 & 0 \end{bmatrix},$

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} -3 & 4 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ -1 & 1 \end{bmatrix};$$

hence

$$x_1 = 5, \quad x_2 = -7, \quad y_1 = -1, \quad y_2 = 1.$$

6. To prove that

$$(A + C) - (A + B) = C - B,$$

note that

$$(a_{ij} + c_{ij}) - (a_{ij} + b_{ij}) = a_{ij} + c_{ij} - a_{ij} - b_{ij} = c_{ij} - b_{ij}.$$

7. No. Although both members of the equation are equal to zero matrices, the orders are not the same.

[pages 17-19]

1-6. Multiplication of a Matrix by a Number

In many texts, the multiplication of a matrix by a number is called multiplication by a scalar. "Scalar" is a term that was introduced in 1853 and was associated with quaternions. Fundamentally, the word "scalar" means a quantity that can be represented on a scale, that is, a real number. The word scalar may well have been introduced to emphasize the two different number systems. In the theory of vectors, "scalar" is used to denote a magnitude in contrast to a vector, which has both magnitude and direction.

It is very important to note that the product of a matrix and a number is a matrix of the same order as the original matrix.

The fundamental properties of multiplication by a number, or by a "scalar," are stated in Theorem 1-4. These properties are worth emphasizing, since they are important in the definition of an algebra.

Exercises 1-6

1. (a) $$2A - B + C = 2\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 5 \\ 6 & 9 & -1 \end{bmatrix} + \begin{bmatrix} 5 & -1 & 0 \\ 7 & 8 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & 1 & -11 \\ 3 & -1 & 8 \end{bmatrix}.$$

(b) $$3A - 4B - 2C = \begin{bmatrix} 6 - 12 - 10 & 3 - 0 + 2 & -9 - 20 - 0 \\ 3 - 24 - 14 & 0 - 36 - 16 & 12 + 4 + 2 \end{bmatrix}$$
$$= \begin{bmatrix} -16 & 5 & -29 \\ -35 & -52 & -18 \end{bmatrix}.$$

(c) $$7A - 2(B - C) = \begin{bmatrix} 14 & 7 & -21 \\ 7 & 0 & 28 \end{bmatrix} - 2\begin{bmatrix} -2 & 1 & 5 \\ -1 & 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 14 & 7 & -21 \\ 7 & 0 & 28 \end{bmatrix} - \begin{bmatrix} -4 & 2 & 10 \\ -2 & 2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 18 & 5 & -31 \\ 9 & -2 & 28 \end{bmatrix}.$$

(d) $$3(A - 2B + 3C) = 3A - 6B + 9C$$
$$= \begin{bmatrix} 6 & 3 & -9 \\ 3 & 0 & 12 \end{bmatrix} - \begin{bmatrix} 18 & 0 & 30 \\ 36 & 54 & -6 \end{bmatrix} + \begin{bmatrix} 45 & -9 & 0 \\ 63 & 72 & -9 \end{bmatrix}$$
$$= \begin{bmatrix} 33 & -6 & -39 \\ 30 & 18 & 9 \end{bmatrix}.$$

[pages 19-23]

2. (a) $2A - B + C = \begin{bmatrix} 5 & 5 & 5 \\ 6 & 1 & -11 \\ 3 & -1 & 8 \end{bmatrix}$.

(b) $3A - 6B + 9C = \begin{bmatrix} 24 & 24 & 24 \\ 33 & -6 & -39 \\ 30 & 18 & 9 \end{bmatrix}$.

(c) $7A - 2(B - C) = \begin{bmatrix} 16 & 16 & 16 \\ 18 & 5 & -31 \\ 9 & -2 & 28 \end{bmatrix}$.

(d) $3(A - 2B + 3C) = \begin{bmatrix} 24 & 24 & 24 \\ 33 & -6 & -39 \\ 30 & 18 & 9 \end{bmatrix}$.

3. $\frac{1}{2}(X + A) = 3(X + (2X + B)) + C,$

$$X + A = 6(X + (2X + B)) + 2C,$$

$$X + A = 6X + 6(2X + B) + 2C,$$

$$X + A = 6X + 12X + 6B + 2C,$$

$$X - 18X = - A + 6B + 2C,$$

$$X = \frac{1}{17}(A - 6B - 2C),$$

$$X = \frac{1}{17}\begin{bmatrix} -24 & -24 & -24 \\ -26 & 3 & -33 \\ -49 & -70 & 12 \end{bmatrix},$$

$$X = \begin{bmatrix} \frac{-24}{17} & \frac{-24}{17} & \frac{-24}{17} \\ \frac{-26}{17} & \frac{3}{17} & \frac{-33}{17} \\ \frac{-49}{17} & \frac{-70}{17} & \frac{12}{17} \end{bmatrix}.$$

4. $2(X + B) = 3\left(X + \frac{1}{2}(X + A)\right) + C,$

$$2X + 2B = 3X + \frac{3}{2}X + \frac{3}{2}A + C,$$

$$4X + 4B = 6X + 3X + 3A + 2C,$$

$$-5X = 3A - 4B + 2C,$$

$$X = \frac{1}{5}\left(-3A + 4B - 2C\right),$$

[pages 23, 24]

$$X = \frac{1}{5}\begin{bmatrix} -2 & -2 & -2 \\ -4 & -1 & 29 \\ 7 & 20 & -14 \end{bmatrix} = \begin{bmatrix} \frac{-2}{5} & \frac{-2}{5} & \frac{-2}{5} \\ \frac{-4}{5} & \frac{-1}{5} & \frac{29}{5} \\ \frac{7}{5} & 4 & \frac{-14}{5} \end{bmatrix}.$$

5. To prove that $x(yA) = (xy)A$: Note that, for real numbers, we have

$$x(ya_{ij}) = xya_{ij} = (xy)a_{ij}.$$

Then apply the definition (Definition 1-2) of matrix equality.

6. To prove that $(x + y)A = xA + yA$: Note that, for real numbers, we have

$$(x + y)a_{ij} = xa_{ij} + ya_{ij}.$$

Then apply the definition (Definition 1-2) of matrix equality.

1-7. Multiplication of Matrices

Many articles and textbooks dealing with matrices describe the operation of multiplication first. It is more interesting than addition and sets a pattern that makes addition seem easy, even dull. In starting addition, which is a most conventional operation, the student is likely to be lead to the easy conclusion that multiplication proceeds in the same simple manner — namely that two matrices of the same order are multiplied together element by corresponding element. That this is not so must be emphasized from the start. In the text, a "practical" problem is presented, one involving television tubes, speakers, and models. A discussion of the problem will help motivate the unusual pattern for multiplication.

If additional motivation is desired at this point, you might tell the class about operations research, a form of scientific work that has grown rapidly during and since the Second World War. In it, scientific methods are applied to the running of businesses, governments, etc., in order to hold production costs to a minimum, to get maximum use from limited resources, etc. An important tool in operations research is linear programming, a new branch of mathematics that makes extensive use of matrices.

Suppose, for instance, that distances in miles from branch automobile factories F_1, F_2, F_3 to towns T_1, T_2, T_3, T_4 are given by the entries in the following table (matrix) of distances:

$$\begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \end{array} \begin{array}{c} \begin{array}{cccc} T_1 & T_2 & T_3 & T_4 \end{array} \\ \begin{bmatrix} 750 & 200 & 300 & 100 \\ 400 & 500 & 250 & 500 \\ 600 & 800 & 400 & 700 \end{bmatrix} \end{array} .$$

The factories produce a total of 1000 identical cars per day:

Production Table

Factory	No. of Cars Produced
F_1	250
F_2	350
F_3	400
Total	1000

and the total daily demand for the cars by the towns is as follows:

Requirement Table

Town	No. of Cars Required
T_1	400
T_2	200
T_3	300
T_4	100
Total	1000

If it costs $1 to ship a car 10 miles, how can the demand be met at minimum total transportation cost?

[pages 24-32]

In the mathematical formulation of the foregoing problem, you have a fine opportunity to drill the class on subscripts and in the use of the Σ notation. Let x_{ij} denote the number of cars shipped daily from factory F_i to town T_j. Then from the production table you see that

$$x_{11} + x_{12} + x_{13} + x_{14} = 250,$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 350,$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 400.$$

These equations can be written more compactly as

$$\begin{aligned} \sum_{j=1}^{4} x_{1j} &= 250, \\ \sum_{j=1}^{4} x_{2j} &= 350, \\ \sum_{j=1}^{4} x_{3j} &= 400. \end{aligned} \tag{1}$$

Similarly, from the demand table you must have

$$\begin{aligned} \sum_{i=1}^{3} x_{i1} &= 400, \\ \sum_{i=1}^{3} x_{i2} &= 200, \\ \sum_{i=1}^{3} x_{i3} &= 300, \\ \sum_{i=1}^{3} x_{i4} &= 100. \end{aligned} \tag{2}$$

Now from the table of distances, and from the fact that it costs \$1 to ship a car 10 miles, the total cost is

$$
\begin{aligned}
C = 75x_{11} &+ 20x_{12} + 30x_{13} + 10x_{14}, \\
+ 40x_{21} &+ 50x_{22} + 25x_{23} + 50x_{24}, \\
+ 60x_{31} &+ 80x_{32} + 40x_{33} + 70x_{34}.
\end{aligned}
$$

The problem is to determine nonnegative integers x_{ij}, subject to the constraints (1) and (2), in such a way as to minimize the cost C. This is a formidable problem for hand computation, but the class may be interested to know that on an electronic computing machine the methods of linear programming would get the answer quite quickly.

Here is a simpler problem of a similar sort that involves matrix multiplication:

A chicken rancher found that certain brands of feed contain the following amounts of vitamins per measure:

$$
\begin{array}{l c}
 & \begin{array}{ccc} \text{Brand I} & \text{Brand II} & \text{Brand III} \end{array} \\
\begin{array}{l} \text{Vitamin A} \\ \text{Vitamin B} \\ \text{Vitamin C} \\ \text{Vitamin D} \end{array} & \begin{bmatrix} 150 & 1000 & 300 \\ 200 & 800 & 300 \\ 700 & 200 & 200 \\ 700 & 800 & 100 \end{bmatrix}
\end{array} .
$$

For a feeding of his flock, the total minimum vitamin requirements were known to be

$$
\begin{array}{l c}
 & \text{Feeding} \\
\begin{array}{l} \text{Vitamin A} \\ \text{Vitamin B} \\ \text{Vitamin C} \\ \text{Vitamin D} \end{array} & \begin{bmatrix} 40{,}000 \\ 40{,}000 \\ 30{,}000 \\ 40{,}000 \end{bmatrix}
\end{array} .
$$

He actually fed the flock in accordance with the following formula showing measures per feeding:

$$
\begin{array}{l c}
 & \text{Feeding} \\
\begin{array}{l} \text{Brand I} \\ \text{Brand II} \\ \text{Brand III} \end{array} & \begin{bmatrix} 20 \\ 30 \\ 40 \end{bmatrix}
\end{array} .
$$

[pages 24-32]

Why did the chickens fail to develop as they should?

When you multiply the vitamin-per-brand matrix by the brand-per-feeding matrix that he used, you get the actual vitamin-per-feeding matrix:

$$\begin{bmatrix} 150 & 1000 & 300 \\ 200 & 800 & 300 \\ 700 & 200 & 200 \\ 700 & 800 & 100 \end{bmatrix} \begin{bmatrix} 20 \\ 30 \\ 40 \end{bmatrix} = \begin{bmatrix} 45,000 \\ 40,000 \\ 28,000 \\ 42,000 \end{bmatrix}.$$

When this is compared with the required vitamin-per-feeding matrix, it is found that the chickens suffered from a slight vitamin C deficiency.

You might now ask the class to adjust the formula in such a way as to get an adequate feeding. For example, it will be found that the formula

$$\begin{array}{l c} & \text{Feeding} \\ \begin{array}{l} \text{Brand I} \\ \text{Brand II} \\ \text{Brand III} \end{array} & \begin{bmatrix} 23 \\ 30 \\ 38 \end{bmatrix} \end{array}$$

is adequate. You might also ask for a computation of the 1×1 cost-per-feeding matrix for various adequate and inadequate feeding formulas, given that the cost-per-brand matrix, measured in cents per measure, is

$$\begin{array}{l c} & \begin{array}{ccc} \text{Brand I} & \text{Brand II} & \text{Brand III} \end{array} \\ \text{Cost} & \begin{bmatrix} 10 & 30 & 20 \end{bmatrix} \end{array}.$$

Other applied problems involving the multiplication of matrices can be found in the book by Kemeny, Snell, and Thompson listed in the Bibliography on page 231 of the text.

There is another kind of problem that can be presented to aid understanding, a problem familiar to students who have studied a considerable amount of trigonometry and analytic geometry. The notion of a mapping occurs early in a student's mathematical training. The concept is used in the discussion of a function. In analytic geometry, the idea is extended when a change of axes is discussed. The formulas for translation are

$$x' = x - h,$$

$$y' = y - k.$$

[pages 24-32]

The more difficult case of rotation of axes involves the formulas

$$x' = \quad x \cos \theta + y \sin \theta,$$
$$y' = - x \sin \theta + y \cos \theta.$$

Consider two mappings A and B that are performed in the order: first B, then A. Let us denote the mappings as follows:

$$B: \quad \begin{aligned} x' &= c_1 x + c_2 y, \\ y' &= d_1 x + d_2 y; \end{aligned}$$

$$A: \quad \begin{aligned} x'' &= a_1 x' + a_2 y', \\ y'' &= b_1 x' + b_2 y'. \end{aligned}$$

To obtain the product mapping AB, we first perform the mapping B and then the mapping A; this gives

$$x'' = a_1(c_1 x + c_2 y) + a_2(d_1 x + d_2 y),$$
$$y'' = b_1(c_1 x + c_2 y) + b_2(d_1 x + d_2 y),$$

and then, by rearranging terms,

$$x'' = (a_1 c_1 + a_2 d_1)x + (a_1 c_2 + a_2 d_2)y,$$
$$y'' = (b_1 c_1 + b_2 d_1)x + (b_1 c_2 + b_2 d_2)y.$$

If the two original coefficient matrices are placed in juxtaposition,

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix},$$

the "row by column" nature of the "product" (AB) coefficients can readily be seen. It can easily be shown also that $AB \neq BA$. You might tell your class that Cayley (1821–1895), the inventor of matrices, proceeded along such a path in his original work on linear transformations in 1858. He was preceded in the study of the algebra of rotations in space by Hamilton (1805–1865).

[pages 24–32]

Once the mechanics of multiplication are introduced, it is important continually to stress the rule, "row by column." Also, it is most helpful to retain in mind the general form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

The subscripts identify the row and column to be multiplied together in order to obtain a particular entry.

The "$\sum$ notation" is so useful that the students should be given considerable practice with it. Without this compact notation, the proof of some theorems becomes very unwieldy.

Through the examples in this section and through the exercises at the end of the section, many students should develop a suspicion that this multiplication is quite different from the multiplication of real numbers. Unless they read ahead of the assignment, it is doubtful if they will speak of the noncommutativity or the divisors of zero. Even a genius such as Hamilton had to face the problem for years before he would admit that AB does not always equal BA! The learning process can be made more exciting if the secret is not immediately revealed to the students and they are allowed to work out their own discovery.

Only about half of the multiplications in Exercises 1-7-4 and 1-7-6 should be assigned to the average class.

Exercises 1-7

1. (a) 2×3.
   (b) 3×3.
   (c) 2×2.
   (d) 4×3.
   (e) 3×2.
   (f) 3×3.
   (g) 4×3.
   (h) 3×3.

2. (a) $\begin{bmatrix} 1\cdot 1 + 2\cdot 2 + 3\cdot 3 + 4\cdot 4 \end{bmatrix} = \begin{bmatrix} 30 \end{bmatrix}$.

   (b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}$.

(c) $\begin{bmatrix} 8 + 18 + 12 & 4 + 3 + 20 \\ -4 - 12 + 0 & -2 - 2 + 0 \end{bmatrix} = \begin{bmatrix} 38 & 27 \\ -16 & -4 \end{bmatrix}$.

(d) $\begin{bmatrix} 8 - 2 & 12 - 4 & 16 + 0 \\ 12 - 1 & 18 - 2 & 24 + 0 \\ 6 - 5 & 9 - 10 & 12 + 0 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 16 \\ 11 & 16 & 24 \\ 1 & -1 & 12 \end{bmatrix}$.

(e) $\begin{bmatrix} 4 + 0 & 8 + 4 & 12 - 2 & 16 + 12 \\ 1 + 0 & 2 + 6 & 3 - 3 & 4 + 18 \end{bmatrix} = \begin{bmatrix} 4 & 12 & 10 & 28 \\ 1 & 8 & 0 & 22 \end{bmatrix}$.

The multiplication

(f) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & -1 & 6 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ is impossible, since the first matrix is 2×4 and the second is 2×2.

3. (a) $5\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 4 \end{bmatrix} = 5\begin{bmatrix} 10 & -10 & 20 \\ 0 & 0 & 0 \\ 2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 50 & -50 & 100 \\ 0 & 0 & 0 \\ 10 & -10 & 20 \end{bmatrix}$.

(b) $15\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \end{bmatrix} = 15\begin{bmatrix} 0 & -1 & -2 \\ 0 & -1 & -2 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -15 & -30 \\ 0 & -15 & -30 \\ 0 & 30 & 60 \end{bmatrix}$.

(c) $5\begin{bmatrix} 2 & -2 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 & -5 & 6 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

$$\begin{aligned} &= 5\begin{bmatrix} 10 - 0 + 4 \end{bmatrix} - \begin{bmatrix} -4 + 5 + 12 \end{bmatrix} \\ &= 5\begin{bmatrix} 14 \end{bmatrix} - \begin{bmatrix} 13 \end{bmatrix} \\ &= \begin{bmatrix} 70 \end{bmatrix} - \begin{bmatrix} 13 \end{bmatrix} \\ &= \begin{bmatrix} 57 \end{bmatrix}. \end{aligned}$$

(d) $\begin{bmatrix} 2 & -2 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 - 0 + 4 \end{bmatrix} + \begin{bmatrix} 0 - 1 + 4 \end{bmatrix} = \begin{bmatrix} 17 \end{bmatrix}$.

(e) $\begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 6 \end{bmatrix} = \begin{bmatrix} 12 & -6 & 36 \\ 2 & -1 & 6 \\ -2 & 1 & -6 \end{bmatrix}$.

(f) $\begin{bmatrix} 2 & -1 & 6 \end{bmatrix}\begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 12 - 1 - 6 \end{bmatrix} = \begin{bmatrix} 5 \end{bmatrix}$.

4. (a) $\begin{bmatrix} 6 & 4 \\ 2 & 0 \end{bmatrix}$.

(b) $\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$.

(c) $\begin{bmatrix} 2r_1 & 2r_2 & 2r_3 \\ 2s_1 & 2s_2 & 2s_3 \\ 2t_1 & 2t_2 & 2t_3 \end{bmatrix}$.

(d) $\begin{bmatrix} \alpha_1 a_1 & \alpha_1 a_2 \\ \alpha_2 b_1 & \alpha_2 b_2 \\ \alpha_3 c_1 & \alpha_3 c_2 \end{bmatrix}$.

(e) $\begin{bmatrix} 0 & 0 & 0 \\ b_1 y_1 & b_1 y_2 & b_1 y_3 \\ 0 & 0 & 0 \end{bmatrix}$.

(f) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 3 & 1 & 1 \end{bmatrix}$.

(g) $\begin{bmatrix} 4 & 7 & 4 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$.

5. $(AB)C = \begin{bmatrix} -3 & -1 & -1 \\ 8 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ 14 \end{bmatrix}$;

$A(BC) = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 7 \end{bmatrix} = \begin{bmatrix} -8 \\ 14 \end{bmatrix}$.

6. (a) $\begin{bmatrix} -7 & 0 & 4 \\ -13 & 0 & 7 \\ -19 & 0 & 10 \end{bmatrix}$.

(b) $\begin{bmatrix} -2 & 2 & 1 \\ -2 & 2 & -2 \\ -2 & 2 & -5 \end{bmatrix}$.

(c) $\begin{bmatrix} 2 & -2 & -3 \\ -2 & 2 & 3 \\ -3 & 3 & 5 \end{bmatrix}$.

(d) $\begin{bmatrix} -11 & 11 & 18 \\ -20 & 20 & 33 \\ -29 & 29 & 48 \end{bmatrix}$.

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(e) $\begin{bmatrix} -11 & 11 & 18 \\ -20 & 20 & 33 \\ -29 & 29 & 48 \end{bmatrix}$.

(f) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} [illegible] & -1 & -3 \\ [illegible] & 0 & 1 \\ [illegible] & 1 & 2 \end{bmatrix} = \begin{bmatrix} -9 & 2 & 5 \\ -15 & 2 & 5 \\ -21 & 2 & 5 \end{bmatrix}$.

(g) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} [illegible] & 1 & 1 \\ [illegible] & 0 & 1 \\ [illegible] & -1 & 0 \end{bmatrix} = \begin{bmatrix} -5 & -2 & 3 \\ -11 & -2 & 9 \\ -17 & -2 & 15 \end{bmatrix}$.

(h) $\begin{bmatrix} -7 & [illegible] & [illegible] \\ -13 & 0 & [illegible] \\ -19 & 0 & 10 \end{bmatrix} - \begin{bmatrix} -2 & 2 & 1 \\ -2 & 2 & -2 \\ -2 & 2 & -5 \end{bmatrix} = \begin{bmatrix} -5 & -2 & 3 \\ -11 & -2 & 9 \\ -17 & -2 & 15 \end{bmatrix}$.

(i) $\begin{bmatrix} 30 & 36 & 42 \\ 66 & 81 & 96 \\ 102 & 126 & 150 \end{bmatrix} - \begin{bmatrix} 3 & 0 & -2 \\ -3 & 0 & 2 \\ -4 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & -3 & -4 \\ 0 & 0 & 0 \\ -2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 30 & 33 & 40 \\ 69 & 81 & 94 \\ 104 & 128 & 150 \end{bmatrix}$.

(j) $\begin{bmatrix} 30 & 36 & 42 \\ 66 & 81 & 96 \\ 102 & 126 & 150 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 468 & 576 & 684 \\ 1062 & 1305 & 1548 \\ 1656 & 2034 & 2412 \end{bmatrix}$.

7. If

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then $AI = A$, $BI = B$, $B^tI = B^t$, $(AI)(I) = AI = A$, $((AI)I)I = A$.

8. $AB = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$, $(AB)^t = \begin{bmatrix} -3 & 6 \end{bmatrix}$;

$B^t = \begin{bmatrix} 1 & [illegible] & [illegible] \end{bmatrix}$, $A^t = \begin{bmatrix} [illegible] & 1 \\ [illegible] & 2 \\ 2 & 1 \end{bmatrix}$,

$B^tA^t = \begin{bmatrix} -3 & 6 \end{bmatrix}$; so $(AB)^t = B^tA^t$.

9. It is given that

the cost-per-part matrix = $\begin{bmatrix} 2 & 3 & 5 \end{bmatrix}_{1\times 3}$,

the parts-per-subassembly matrix = $\begin{bmatrix} 4 & 1 \\ 3 & 5 \\ 7 & 2 \end{bmatrix}_{3\times 2}$,

the subassembly-per-model matrix = $\begin{bmatrix} 2 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}_{2\times 3}$,

[pages 34, 35]

and the model-per-day matrix = $\begin{bmatrix} 7 & 8 & 8 \\ 3 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}_{3 \times 3}$.

Then

$$\text{the parts-per-model matrix} = \begin{bmatrix} 4 & 1 \\ 3 & 5 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 8 & 13 \\ 21 & 23 & 31 \\ 20 & 15 & 24 \end{bmatrix},$$

and the cost-per-day matrix

$$= \begin{bmatrix} 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 11 & 8 & 13 \\ 21 & 23 & 31 \\ 20 & 15 & 24 \end{bmatrix} \begin{bmatrix} 7 & 8 & 8 \\ 3 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 185 & 160 & 239 \end{bmatrix} \begin{bmatrix} 7 & 8 & 8 \\ 3 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2492 & 3315 & 3714 \end{bmatrix} .$$

1-8. Properties of Matrix Multiplication

Now that the definition of matrix multiplication has been given, its properties must be revealed, but slowly.

Although $AB \neq BA$ in general, there are many illustrations where $AB = BA$. The class can be put to work discovering examples. A contest can be devised with a prize to the student who brings in the largest number of illustrations.

A similar contest can be established around divisors of zero. Who can bring in the largest number of illustrations of $AB = \underline{0}$ when $A \neq \underline{0}$ and $B \neq \underline{0}$?

Although some texts use the same notation for the real number 0 and the matrix $\underline{0}$, this distinction should be emphasized, particularly at the beginning, since it emphasizes two systems, the real numbers and the matrices.

Only about half of the multiplications in Exercises 1, 2, and 5 should be assigned to the average class.

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Exercises 1–8

1. (a) $AB = \begin{bmatrix} -1 & 2 \\ -1 & 4 \end{bmatrix}$, (b) $BA = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$,

(c) $(AB)A = \begin{bmatrix} 5 & 6 \\ 11 & 14 \end{bmatrix}$, (d) $(BA)A = \begin{bmatrix} 7 & 10 \\ 8 & 12 \end{bmatrix}$,

(e) $(BA)B = \begin{bmatrix} -1 & 2 \\ 0 & 2 \end{bmatrix}$, (f) $B(BA) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$,

(g) $A(AB) = \begin{bmatrix} -3 & 10 \\ -7 & 22 \end{bmatrix}$, (h) $((BA)A)B = \begin{bmatrix} -3 & 10 \\ -4 & 12 \end{bmatrix}$,

(i) $((AB)A)B = \begin{bmatrix} -1 & 6 \\ -3 & 14 \end{bmatrix}$.

2. (a) $\begin{bmatrix} 2 & 3 & -2 \\ 5 & 6 & -5 \\ 8 & 9 & -8 \end{bmatrix}$, (b) $\begin{bmatrix} -6 & -6 & -6 \\ 6 & 6 & 6 \\ -2 & -1 & 0 \end{bmatrix}$,

(c) $\begin{bmatrix} 0 & 3 & 6 \\ -6 & 0 & 6 \\ -12 & -3 & 6 \end{bmatrix}$, (d) $\begin{bmatrix} -72 & -90 & -108 \\ 72 & 90 & 108 \\ -6 & -9 & -12 \end{bmatrix}$,

(e) $\begin{bmatrix} -6 & -6 & 6 \\ 6 & 6 & -6 \\ -1 & 0 & 1 \end{bmatrix}$, (f) $\begin{bmatrix} -4 & -5 & -6 \\ 4 & 5 & 6 \\ 2 & 1 & 0 \end{bmatrix}$,

(g) $\begin{bmatrix} 36 & 42 & -36 \\ 81 & 96 & -81 \\ 126 & 150 & -126 \end{bmatrix}$, (h) $\begin{bmatrix} -90 & -108 & 90 \\ 90 & 108 & -90 \\ -9 & -12 & 9 \end{bmatrix}$,

(i) $\begin{bmatrix} 3 & 6 & -3 \\ 0 & 6 & 0 \\ -3 & 6 & 3 \end{bmatrix}$.

3. $AI = IA = A$, $BI = IB = B$, $(AI)B = AB = \begin{bmatrix} 2 & 3 & -2 \\ 5 & 6 & -5 \\ 8 & 9 & -8 \end{bmatrix}$.

4. (a) $(A + B)(A + B) = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 6 & 15 \end{bmatrix}$;

$$A^2 = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix},$$

$$B^2 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix},$$

$$2AB = 2\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = 2\begin{bmatrix} 0 & -2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 4 & 8 \end{bmatrix};$$

$$A^2 + 2AB + B^2 = \begin{bmatrix} 1+0+1 & -3-4+0 \\ 0+4+3 & 4+8+4 \end{bmatrix} = \begin{bmatrix} 2 & -7 \\ 7 & 16 \end{bmatrix} \neq \begin{bmatrix} 3 & -6 \\ 6 & 15 \end{bmatrix}.$$

Therefore, $A^2 + 2AB + B^2 \neq (A + B)(A + B)$.

(b) $(A + B)(A - B) = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -4 & -1 \end{bmatrix}$;

$$A^2 - B^2 = \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & -2 \\ -4 & -1 \end{bmatrix}.$$

Therefore, $A^2 - B^2 \neq (A + B)(A - B)$.

5. $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$,

$$A^3 = A(A^2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 27 \end{bmatrix},$$

$$B^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix},$$

$$B^3 = B(B^2) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix},$$

$$AB^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 24 \end{bmatrix},$$

$$A^2B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 18 \end{bmatrix}.$$

6. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$,

$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

There are many more.

7. $A^2 = \begin{bmatrix} 0\cdot0 + 0\cdot1 & 0\cdot0 + 0\cdot0 \\ 1\cdot0 + 0\cdot1 & 1\cdot0 + 0\cdot0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \underline{0}$.

Any 2×2 matrix that has an arbitrary entry at any one of the 4 positions

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while the other 3 entries are zero will satisfy this equation; thus, if

$$A = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}, \quad \text{then} \quad A^2 = \begin{bmatrix} 0\cdot 0 + x\cdot 0 & 0\cdot x + x\cdot 0 \\ 0\cdot 0 + 0\cdot 0 & 0\cdot x + 0\cdot 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \underline{0}.$$

Also, among others, $A = \begin{bmatrix} x & x \\ -x & -x \end{bmatrix}$ satisfies the equation $A^2 = \underline{0}$. (See Exercise 1-9-6.)

8. $$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad AA = A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad AA^2 = A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

1-9. Properties of Matrix Multiplication (Concluded)

It should be emphasized that examples, no matter how numerous, do not prove a general law concerning an infinitude of cases. Although the text has many examples illustrating the associative law, the right-hand distributive law, and the left-hand distributive law, proofs are still necessary if we are to state, "We have proved the law." The proofs, which involve the summation or sigma notation, are rather difficult and may be beyond some classes. Certainly, the theorems can be demonstrated on an intuitive basis.

There is no difficulty in presenting the ideas associated with the zero matrices,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and the unit matrices,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The following question can be presented to the class, "Why isn't the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

a unit matrix?"

Exercises 1-9

1. (a) $A(B + C) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$,

$AB + AC = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = A(B + C)$.

(b) $(B + C)A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$,

$BA + CA = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = (B + C)A$.

(c) $A(B + C) = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$,

$AB + CA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \neq A(B + C)$.

(d) $A(B + C) = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$,

$BA + CA = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \neq A(B + C)$.

2. $AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq \underline{0}$, $\quad BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \underline{0}$.

3. $AB = \begin{bmatrix} ac - bd & ad + bc \\ -bc - ad & -bd + ac \end{bmatrix} = \begin{bmatrix} ac - bd & ad + bc \\ -(ad + bc) & ac - bd \end{bmatrix}$,

$BA = \begin{bmatrix} ca - db & cb + da \\ -ad - bc & -db + ac \end{bmatrix} = \begin{bmatrix} ac - bd & ad + bc \\ -(ad + bc) & ac - bd \end{bmatrix} = AB$.

If, for example, $a = 2$, $b = 3$, $c = 1$, $d = 4$, then

$A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$, $\quad B = \begin{bmatrix} 1 & 4 \\ -4 & 1 \end{bmatrix}$,

$AB = \begin{bmatrix} -10 & 11 \\ -11 & -10 \end{bmatrix}$, and $BA = \begin{bmatrix} -10 & 11 \\ -11 & -10 \end{bmatrix} = AB$.

4. If $\begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -x & -14x & 7x \\ 0 & 1 & 0 \\ x & 4x & -2x \end{bmatrix} = I$,

then

$$\begin{aligned} -2x + 0 + 7x &= 1, \\ -14x + 2 + 4x &= 0, \\ 7x - \quad 2x &= 1, \end{aligned}$$

each of which is equivalent to $x = 1/5$. The other six equations are

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satisfied for all x, so there is exactly one x, namely $x = 1/5$, that satisfies all nine equations.

5. $$AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underline{0} = BA.$$

$$AC = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} = CA.$$

$$BC = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underline{0} = CB.$$

6. $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad AA = A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$AA^2 = A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Other solutions are $A = I$ and

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

7. $$A^2 = \begin{bmatrix} (ab)(ab) + (b^2)(-a^2) & (ab)b^2 + b^2(-ab) \\ (-a^2)(ab) + (-ab)(-a^2) & (-a^2)(b^2) + (-ab)(-ab) \end{bmatrix}$$

$$= \begin{bmatrix} a^2b^2 - a^2b^2 & ab^3 - ab^3 \\ -a^3b + a^3b & -a^2b^2 + a^2b^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

For example, let $a = 2$, $b = 3$. Then

$$A = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 36 - 36 & 54 - 54 \\ -24 + 24 & -36 + 36 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

8. (a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(b) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(d) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

(e) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(f) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

D and E, D and F, and E and F are anticommutative.

9. $A^2 = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$,

$-5A = \begin{bmatrix} -15 & -5 \\ 5 & -10 \end{bmatrix}$,

$7I = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$.

Adding, we obtain $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; thus A satisfies the equation

$$A^2 - 5A + 7I = \underline{0}.$$

10. $(A + B)(A - B) = A^2 + BA - AB - B^2$;

but in general $BA \neq AB$, so the middle terms do not give the zero matrix when combined.

For example, if

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix},$$

then

$$AB = \begin{bmatrix} 3 & 3 \\ 3 & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} -2 & -1 \\ 1 & 5 \end{bmatrix},$$

so that $AB \neq BA$, and therefore in this case

$$(A + B)(A - B) \neq A^2 - B^2.$$

In fact,

$$(A + B)(A - B) = \begin{bmatrix} -9 & 0 \\ -4 & 1 \end{bmatrix},$$

while

$$A^2 - B^2 = \begin{bmatrix} -4 & 4 \\ -2 & -4 \end{bmatrix}.$$

But (see Exercise 1-9-3) if

$$A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad \text{then } AB = BA,$$

so that, in this case,

$$(A + B)(A - B) = A^2 - B^2.$$

11. Let $V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad W = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}.$

Then

$$V^t = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}, \quad W^t = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix},$$

so that

$$V^tW = \sum_{i=1}^{n} v_i w_i = W^tV.$$

The class will understand this better if you illustrate it for $n = 2$ and $n = 3$.

12. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{m \times p}, \quad B = \begin{bmatrix} b_{jk} \end{bmatrix}_{p \times n}.$

Then

$$AB = \begin{bmatrix} \sum_{j=1}^{p} a_{ij} b_{jk} \end{bmatrix}_{m \times n},$$

$$\uparrow$$

entry in position (i,k)

so that

$$(AB)^t = \begin{bmatrix} \sum_{j=1}^{p} a_{ij} b_{jk} \end{bmatrix}_{n \times m};$$

$$\uparrow$$

entry in position (k,i)

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further

$$B^t = \left[b_{jk}\right]_{n \times p}, \qquad A^t = \left[a_{ij}\right]_{p \times m},$$

entry in position (k,j) $\qquad$ entry in position (j,i)

so that

$$B^tA^t = \left[\sum_{j=1}^{p} b_{jk}a_{ij}\right]_{n \times m}.$$

entry in position (k,i)

Hence $(AB)^t = B^tA^t$.

13. Let $B = \left[b_{ij}\right]_{m\times p}$, $C = \left[c_{ij}\right]_{m\times p}$, and $A = \left[a_{jk}\right]_{p\times n}$.

Then

$$B + C = \left[b_{ij} + c_{ij}\right]_{m\times p},$$

$$(B + C)A = \left[\sum_{j=1}^{p} (b_{ij} + c_{ij})a_{jk}\right]_{m\times n}$$

$$= \left[\sum_{j=1}^{p} (b_{ij}a_{jk} + c_{ij}a_{jk})\right]_{m\times n}$$

$$= \left[\sum_{j=1}^{p} b_{ij}a_{jk} + \sum_{j=1}^{n} b_{ij}a_{jk}\right]_{m\times n}$$

$$= \left[\sum_{j=1}^{n} b_{ij}a_{jk}\right]_{m\times n} + \left[\sum_{j=1}^{n} c_{ij}a_{jk}\right]_{m\times n}$$

$$= BA + CA.$$

1-10. Summary

The summary of this chapter recalls the principal results thus far obtained; it also points toward the developments of Chapter 2 and 3. It recalls some differences between matrix algebra and elementary algebra; it also points toward yet another difference.

You should dwell with the class on the fact that the operations of subtraction and division are inverse to the basic operations of addition and multiplication.

Every matrix A has a negative, or "additive inverse" $-A$; consequently, the problem of subtraction is always solvable for matrices that are conformable for addition. This statement is almost too trivial to be understood, and it should be thoroughly illustrated. Thus, to solve the equation

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} + X = \begin{bmatrix} 3 \\ 7 \end{bmatrix},$$

or

$$A + X = B,$$

for X, we add the negative of A to each side of the equation, getting

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} + X = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 7 \end{bmatrix},$$

whence

$$X = \begin{bmatrix} 1 \\ 8 \end{bmatrix} = B - A.$$

Analogous statements cannot be made concerning the problem of division in matrix algebra. Thus the problem of solving

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

or

$$AX = B,$$

for X has no solution X. You can see this by letting

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$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and trying to solve for a, b, c, and d. Thus if there were a solution, then we would have

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

an impossibility since $0 \neq 1$.

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Chapter 2

THE ALGEBRA OF 2×2 MATRICES

2-1. Introduction

Much more real mathematics is met in this chapter than in the preceding one. Chapter 1 is largely concerned with manipulation and depends on computational skill; even the student without much mathematical insight can handle the material very readily. In Chapter 2, more mathematical intuition is needed, and the ideas presented are much more subtle. Extensive discussion about and around the ideas is necessary in order to convey to the pupils the true considerations that are involved. Since there are few routine problems in the exercises at the end of each section in Chapter 2, only one or two of the exercises should be assigned at a time. It is better that the student should do fewer, but more thought-provoking, problems than that he should do a larger number that are merely mechanical.

For a class whose mathematical ability is such that the students experience difficulty with Chapter 1, it may be best to omit Chapter 2 and proceed directly to Chapter 3. On the other hand, it is very likely that Chapter 2 will prove most interesting and challenging to any class with high mathematical ability.

Of all subsets of rectangular matrices, probably the most interesting is the set of 2×2 matrices. There are many mathematicians who feel that the 2×2 matrices have inherently enough value in themselves and can be so elegantly discussed that they alone should be presented in a text designed for secondary schools. The word "elegant" is one reserved by mathematicians for special situations. If the proof of a significant theorem is concisely and cleverly presented, it is called "elegant." If the exposition of a difficult mathematical concept is lucidly and originally done, it is said to be "elegant." This adjective is seldom used, for it confers high distinction. The parts of the present text that would, relatively, "rate" this accolade are probably Chapter 2 and the Appendix.

The transition from Chapter 1 to Chapter 2 is built around the multiplicative inverse. A full discussion of inverses for matrices of arbitrary order is beyond the scope of the book. The problem can be fully handled, however, if we confine our attention to the relatively simple subset of 2×2 matrices.

The general method of determining the inverse, if it exists, is approached slowly. As in Chapter 1, the learning process is made as much an adventure as possible. Proceed slowly, demanding that the students wrestle with the ideas and work out their own solutions if possible.

The purpose of the exercises in the present section is to illustrate the verification (or falsification), in a variety of unusual situations, of the properties occurring in the definitions of ring and field. If a property is valid, a reason should be given; if the property does not hold, a counter-example is called for.

Exercises 2-1

1. (a) The set of integers is closed under addition; that is, any two integers can be added, and their sum is an integer.

   (b) The set of even numbers is closed under multiplication.

   (c) The set $\{1\}$ is closed under multiplication.

   (d) The set of positive irrational numbers is not closed under division. (For example, $\sqrt{2}/\sqrt{2} = 1$, which is not irrational.)

   (e) The set of integers is closed under the operation of squaring.

   (f) The set of numbers $A = \{x: x \geq 3\}$ is closed under addition. (This is read " x such that x is greater than or equal to 3.")

2. (a) False.
   (b) False.
   (c) True, commutative.
   (d) True, commutative.
   (e) False.
   (f) True, commutative.
   (g) True, commutative.

3. (a) Is not commutative.
   (b) Is commutative.
   (c) Is commutative.
   (d) Is not commutative.
   (e) Is not commutative.
   (f) Is commutative.

4. (a) Is not associative.
   (b) Is associative.
   (c) Is not associative.
   (d) Is associative.
   (e) Is not associative.
   (f) Is not associative.

5. (a) Is distributive. In arithmetic, multiplication is distributive over addition.

   (b) Is distributive. For example,

$$2 * (3 \mp 4) = 2 * 14 = 14,$$

   and

$$(2 * 3) \mp (2 * 4) = 3 \mp 4 = 14.$$

   (c) Is not distributive. For example,

$$2 * (3 \mp 4) = 2 * 8 = 16,$$

   while

$$(2 * 3) \mp (2 * 4) = 6 \mp 8 = 15.$$

   The answers are the same for left-hand distribution as they are for right-hand distribution because the particular operations (*) in (a), (b), and (c) are commutative.

6. (a) No. (Additive identity and additive inverses are lacking.)

   (b) Yes. The class should check that each of the field properties is satisfied.

   (c) No. (For example, multiplicative inverse of 2 is lacking.)

   (d) Yes. Again, the class should check that each of the field properties is satisfied.

2-2. The Ring of 2×2 Matrices

There is an important distinction between a field and a ring. Every field is a ring, but the converse statement is not true. For classes that have a strong mathematical background, there are many examples both of fields and of rings. The set of rationals, the set of reals, and the set of complex numbers are all fields under the usual addition and multiplication. Perhaps the best and simplest examples of rings that are not fields are the ones that occur in

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the finite number systems. Numbers modulo 4 form a ring, as do all finite number systems for which the modulus is not a prime. A very simple illustration of a ring that is not a field is furnished by the infinite set

$$\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$$

and its properties with respect to addition and multiplication. Note that in the general definition of a ring we assume neither the commutativity of multiplication, nor the existence of a multiplicative identity element, nor the existence of a multiplicative inverse.

In determining whether any set under certain operations fulfills the criteria for a ring, it is important to prove carefully that each postulate is satisfied. With beginners there is a tendency to dismiss any proof with the word "obvious." This observation is not testimony to their indifference or casualness, but rather testimony to their lack of appreciation of the subtleties of the proof. In the particular case of 2×2 matrices, the inexperienced student is apt to dismiss the step-by-step demonstration that the postulates are satisfied, since he feels that the subset must satisfy the same criteria as the superset itself. There are many examples that can upset any such notion. Through their previous experience with mathematics, most students are quite aware that a counterexample can prove that certain propositions are not valid. This time, it is important to know that a thousand examples are no proof of the validity of any general proposition. A general proof that covers <u>all</u> cases is necessary. In order to prove the propositions in the exercises, it might be desirable to hold the students to a certain pattern so that they will systematically cover each postulate. In order to prove that a certain set under specific operations is <u>not</u> a ring, it is sufficient to exhibit, in the set, an example for which at least one postulate fails to hold.

At all times, however, you should avoid overwhelming the class with details, such as memorizing the definition of a ring, so that the students will not lose sight of the larger objectives of the chapter. From time to time the classwork should be interrupted in order to review these latter goals.

Exercises 2-2

1. The set of integers is closed, commutative, and associative under addition, and there are identity and inverse elements for addition; further, the

[pages 57-61]

set is closed and associative under multiplication, and multiplication distributes over addition in both directions. Hence, the set of integers under addition and multiplication is a ring.

2. (a) Is a ring.

   (b) Is not a ring. (For example, $1 + 1$ is not in the set.)

   (c) Is not a ring. (For example, $\frac{1}{2} \times \frac{1}{2}$ is not in the set.)

3. Is a ring.

4. Is not a ring. For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix},$$

so that the set is not closed under addition.

2-3. The Uniqueness of the Multiplicative Inverse

At the secondary-school level, the problem of the multiplicative inverse is ordinarily discussed as the problem of division. Indeed, continual reference is made to the four operations of addition, subtraction, multiplication, and division as if they were on an equal footing. As a result, students do not have a clear understanding of the operations. Subtraction and division should not be introduced as independent operations, but rather as the inverses of addition and multiplication, respectively. This idea is having a considerable amount of influence on the newer ninth- and tenth-grade texts, where less emphasis is being placed on the operations of subtraction and division as such, and more on the role of the additive inverse and the multiplicative inverse. At the moment, however, we are dealing principally with students who have a concept of four operations. In order to clarify the relationship between multiplication and division, much time should be spent on the subtleties of the multiplicative inverse. Since the exercises in the preceding section serve as an introduction, each of these exercises should be reviewed. The general discussion of multiplicative inverses should begin with the real number system. The first example should involve the integers under multiplication. The integers, except ± 1, do not have integers as multiplicative inverses. For example, for the integer 2 there is no integer x such that $2x = 1$.

The number 1 is the identity element for multiplication in the real number system. The identity element for multiplication in the set of 2×2 matrices is the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

the symbol for which is I. Given a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we call the matrix

$$B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

the inverse of A if

$$AB = I = BA.$$

The question of uniqueness seldom arises in the traditional secondary-school mathematics course. This is a regrettable omission. Before any degree of mathematical maturity can be achieved, it is necessary to understand that <u>existence</u> and <u>uniqueness</u> are two distinct notions. Unfortunately, there are few opportunities to introduce the subject in the ordinary secondary-school curriculum. Before introducing Theorem 2-2, it is important to stress the significance of "uniqueness." Perhaps this simple example will help the class to understand the point: There <u>exists</u> a positive integer less than 3, but <u>not</u> a <u>unique</u> one; there <u>exists</u> a positive integer less than 2, and it is <u>unique</u>; there does <u>not exist</u> any positive integer less than 1.

<u>Exercises 2-3</u>

1. (a) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix},$

   and thus there are no values a, b, c, d that yield I, since $0 \neq 1$.

[pages 62-68]

(b) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a+b & 0 \\ c+d & 0 \end{bmatrix} \neq I,$

since $0 \neq 1$.

(c) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix} \neq I,$

since $a + b$ would have to be both 1 and 0 for equality to hold.

(d) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} -3b & 0 \\ -3d & 0 \end{bmatrix} \neq I,$

since $0 \neq 1$.

2. (a) An inverse pair, since

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) An inverse pair, since

$$\begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(c) Not an inverse pair.

(d) Not an inverse pair.

(e) An inverse pair if $ad - bc = 1$;
not an inverse pair if $ad - bc \neq 1$.

3. Let

$$A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}.$$

Then $AB = \underline{0}$. If A had an inverse A^{-1}, then we would have

$$\underline{0} = A^{-1}\underline{0} = A^{-1}(AB) = (A^{-1}A)B = IB = B,$$

so that $B = \underline{0}$, which is not so. Hence, A does not have an inverse. A similar argument proves that B does not have an inverse.

4. $$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}^2 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} a^2 + bc & 0 \\ 0 & a^2 + bc \end{bmatrix}.$$

Hence, if $a^2 + bc = 0$, then

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}^2 = \underline{0}.$$

The argument of Exercise 3 may now be used to prove that if $a^2 + bc = 0$ then

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

has no inverse. If it did have an inverse M, we could use M as a left multiplier to obtain

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \underline{0},$$

which has no inverse.

5. Same argument as that for Exercise 3. Suppose A has an inverse, A^{-1}. Then

$$\underline{0} = A^{-1}\underline{0} = A^{-1}(AB) = (A^{-1}A)B = IB = B.$$

Therefore we would have $B = \underline{0}$, contradicting the hypothesis that $B \neq \underline{0}$. The matrix B can have an inverse only if $A = \underline{0}$, since if B has an inverse B^{-1} then

$$\underline{0} = \underline{0}B^{-1} = (AB)B^{-1} = A(BB^{-1}) = AI = A.$$

For example, if $A = \underline{0}$ and $B = I$, then the conditions are satisfied and B has an inverse.

6. $A^2 - 4A = A(A - 4I) = \underline{0}$, by hypothesis. If A has an inverse, A^{-1}, then upon multiplying the members of the equation on the left by A^{-1}, we get $A - 4I = \underline{0}$, or $A = 4I$, one of the possibilities. The other possibility is that A does not have an inverse.

7. If $AB = I = CA$, then $B = IB = (CA)B = C(AB) = CI = C$.

8. $\begin{bmatrix} 1 & 4 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 2 & -4 \\ 0 & -6 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

9. $\begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix} = I,$

$$\begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}^2 = \begin{bmatrix} 7 & -18 \\ -12 & 31 \end{bmatrix},$$

$$\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}^2 = \begin{bmatrix} 31 & 18 \\ 12 & 7 \end{bmatrix},$$

$$\begin{bmatrix} 7 & -18 \\ -12 & 31 \end{bmatrix}\begin{bmatrix} 31 & 18 \\ 12 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 31 & 18 \\ 12 & 7 \end{bmatrix}\begin{bmatrix} 7 & -18 \\ -12 & 31 \end{bmatrix} = I,$$

so the squares are inverses.

More generally, if $AB = I$, then

$$(A^2)(B^2) = A(AB)B = AIB = AB = I.$$

Similarly if $BA = I$, then $B^2A^2 = I$. Thus we have shown that if A and B are inverses of one another, then so are A^2 and B^2.

As for transposes, by Exercise 1-9-11 we have

$$I = I^t = (AB)^t = B^tA^t; \text{ also, } I = I^t = (BA)^t = A^tB^t.$$

For the particular example in the text, we have

$$A^t = \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix} \text{ and } B^t = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix},$$

whence matrix multiplication gives

$$A^tB^t = \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$B^tA^t = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

10. (a) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

so

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(b) $$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

(c) $$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

so by (a) we have

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{4} = I.$$

Hence,

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{3} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

11. If $\theta = 120^\circ$, then $\cos\theta = -1/2$ and $\sin\theta = \sqrt{3/2}$, so

$$B = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \quad B^2 = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \quad B^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The result can also be obtained trigonometrically from the expressions

$$B^2 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix},$$

$$B^3 = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta \cos \theta - \sin 2\theta \sin \theta & \cos 2\theta \sin \theta + \sin 2\theta \cos \theta \\ -\sin 2\theta \cos \theta - \cos 2\theta \sin \theta & -\sin 2\theta \sin \theta + \cos 2\theta \cos \theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta \end{bmatrix},$$

and the values $\cos 240^\circ = -1/2$, $\sin 240^\circ = -\sqrt{3}/2$, $\cos 360^\circ = 1$, $\sin 360^\circ = 0$.

12. $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, $A^2 = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$,

$$A^2 - 2A + I = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} -6 & 8 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The transpose of A also satisfies the equation, since

$$(A^t)^2 - 2A^t + I^t = (A^2 - 2A + I)^t = \underline{0}^t = \underline{0}.$$

13.
$$pA^2 + qA = -rI,$$

$$-\frac{p}{r}A^2 - \frac{q}{r}A = I,$$

so

$$A\left(-\frac{p}{r}A - \frac{q}{r}I\right) = I,$$

and

$$\left(-\frac{p}{r}A - \frac{q}{r}I\right)A = I.$$

14.
$$X^2 = \begin{bmatrix} p & q \\ r & s \end{bmatrix}^2 = \begin{bmatrix} p^2 + qr & pq + qs \\ pr + rs & qr + s^2 \end{bmatrix},$$

whence

$$X^2 - (p + s)X + (ps - qr)I$$

$$= \begin{bmatrix} p^2 + qr & pq + qs \\ pr + rs & qr + s^2 \end{bmatrix} + \begin{bmatrix} -p^2 - ps & -pq - qs \\ -pr - rs & -ps - s^2 \end{bmatrix}$$

[pages 70, 71]

$$+ \begin{bmatrix} ps - qr & 0 \\ 0 & ps - qr \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} .$$

Using the technique of 13, if $ps - qr \neq 0$, we obtain

$$X \left[\frac{-X}{ps - qr} + \frac{(p + s)I}{ps - qr} \right] = I,$$

$$\left[\frac{-X}{ps - qr} + \frac{(p + s)I}{ps - qr} \right] X = I,$$

so that

$$X^{-1} = \frac{-X}{ps - qr} + \frac{(p + s)I}{ps - qr} .$$

Thus if $ps - qr \neq 0$, then X^{-1} exists.

If $ps - qr = 0$, then

$$X^2 - (p + s)X = \underline{0},$$

or

$$X \left[X - (p + s)I \right] = \underline{0}.$$

Hence, if X^{-1} exists then left-hand multiplication by X^{-1} yields $X - (p + s)I = \underline{0}$, or

$$X = \begin{bmatrix} p + s & 0 \\ 0 & p + s \end{bmatrix} .$$

But by hypothesis,

$$X = \begin{bmatrix} p & q \\ r & s \end{bmatrix} .$$

Equating entries, we obtain $0 = q = r$. Also, as $p + s = p$ and $p + s = s$, we have $p = s = 0$. But then $X = \underline{0}$ and we know $\underline{0}$ does not have an inverse. Hence the assumption that X^{-1} exists leads to a contradiction, so that if $ps - qr = 0$ then X^{-1} does not exist. In other words, if X^{-1} exists, then $ps - qr \neq 0$.

[pages 70, 71]

15. If $X^2 = \underline{0}$, then, by Exercise 14,

$$-(p + s)X + (ps - qr)I = \underline{0},$$

or

$$(p + s)X = (ps - qr)I.$$

It follows that if $p + s = 0$ then $ps - qr = 0$ and we are through. If we should have $p + s \neq 0$, then from

$$\begin{bmatrix} (p + s)p & (p + s)q \\ (p + s)r & (p + s)s \end{bmatrix} = \begin{bmatrix} ps - qr & 0 \\ 0 & ps - qr \end{bmatrix},$$

we would obtain $q = r = 0$, whence

$$(p + s)p = ps = (p + s)s,$$

or $p = s = 0$, a contradiction of $p + s \neq 0$.

2-4. The Inverse of a Matrix of Order Two

This section can be considered the kernel of Chapter 2. Do not rush through the earlier part of the chapter, however, since the discovery process would then be left uncultivated and much valuable mathematics would be slighted.

The method that is used in obtaining a general expression for the inverse of a nonsingular $(ad - bc \neq 0)$ 2×2 matrix is quite simple. As far as understanding goes, the most difficult paragraph is perhaps the three-line one on pages 73 and 74 of the text.

Note that in the development of a general formula for the inverse, our work first involves the right-hand inverse; namely, given a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we find

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} \text{ such that } \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = I.$$

In order to satisfy our definition, namely that B is the inverse of A if

$$AB = I = BA,$$

we must then demonstrate that the right-hand inverse is also the left-hand inverse. After this has been demonstrated, it is possible to state the results as a theorem. It is again important to emphasize that the converse of any particular theorem does not necessarily follow from the theorem itself; in fact, the converse might not even be valid. There are many familiar examples, particularly in geometry, that can be used to clarify this point.

Exercises 2-4

1. (a) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

   (b) Inverse does not exist, since $h = ad - bc = 0$.

   (c) $\begin{bmatrix} \frac{21}{h} & -\frac{7}{h} \\ -\frac{9}{h} & -\frac{3}{h} \end{bmatrix}$, where $h = -126$.

   (d) $\begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2} & 2 \end{bmatrix}$.

   (e) $\begin{bmatrix} -\frac{1}{2} & 0 \\ \frac{3}{8} & \frac{1}{4} \end{bmatrix}$.

   (f) $\begin{bmatrix} \frac{1}{2} & \frac{a}{14} \\ 0 & -\frac{1}{7} \end{bmatrix}$.

   (g) Inverse does not exist, since $h = ad - bc = 6 - 6 = 0$.

2. We have already established that a 2×2 matrix has no inverse if and only if $h = ad - bc = 0$. We now find the values of x for which $h = 0$:

[pages 71-76]

(a) $x^3 - 1 = 0$, $x^3 = 1$, $x = 1$ (since $x \in R$).

(b) $x^3 = 0$, $x = 0$.

(c) $(x + 2)(x - 1) = 0$, $x \in \{-2, 1\}$.

(d) $3x^2 - 2(x - 1) = 0$,

$3x^2 - 2x + 2 = 0$.

Discriminant is negative; hence there is no real solution and an inverse exists for all $x \in R$.

3. (a) $$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

sin $\cos^2\theta + \sin^2\theta = 1$.

(b) $$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\alpha - \sin\theta\sin\alpha & \cos\theta\sin\alpha + \sin\theta\cos\alpha \\ -\sin\theta\cos\alpha - \cos\theta\sin\alpha & -\sin\theta\sin\alpha + \cos\theta\cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha + \theta) & \sin(\alpha + \theta) \\ -\sin(\alpha + \theta) & \cos(\alpha + \theta) \end{bmatrix}.$$

4. Suppose A has an inverse, say B. Then

$$AB = BA = I,$$

so that

$$(AB)^t = (BA)^t = I^t = I,$$

or

$$B^tA^t = A^tB^t = I.$$

Hence A^t has B^t as its inverse.

Conversely, suppose A^t has an inverse, say B^t. Then

$$A^tB^t = B^tA^t = I,$$

so that

$$(A^tB^t)^t = (B^tA^t)^t = I^t = I,$$

or

$$(B^t)^t(A^t)^t = (A^t)^t(B^t)^t = I,$$

or

$$BA = AB = I.$$

Hence A also has an inverse, namely, B.

We have shown, above, that if A has an inverse B, then

$$B^tA^t = A^tB^t = I.$$

Hence, as $B = A^{-1}$, we have

$$(A^{-1})^tA^t = A^t(A^{-1})^t = I.$$

But this says that the transpose of A^{-1} is the inverse of A^t since the product in both directions is I.

5. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{h} & -\frac{b}{h} \\ -\frac{c}{h} & \frac{a}{h} \end{bmatrix}, \quad h = ad - bc \neq 0.$

Since

$$\left(\frac{d}{h}\right)\left(\frac{a}{h}\right) - \left(\frac{-b}{h}\right)\left(\frac{-c}{h}\right) = \frac{ad - bc}{h^2} = \frac{h}{h^2} = \frac{1}{h} \neq 0,$$

A^{-1} has an inverse given by

$$\left[\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\right]^{-1} = \begin{bmatrix} \frac{d}{h} & -\frac{b}{h} \\ -\frac{c}{h} & \frac{a}{h} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{ah}{h} & \frac{bh}{h} \\ \frac{ch}{h} & \frac{dh}{h} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

6. If $A \in M$ has an inverse, say A^{-1}, then the solution of $AX = B$, for $B \in M$, may be indicated as follows:

$$AX = B,$$

$$A^{-1}(AX) = A^{-1}B,$$

or

$$X = A^{-1}B.$$

(a) $$\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{4}{11} & -\frac{3}{11} \\ \frac{1}{11} & \frac{2}{11} \end{bmatrix}$$

$$\begin{bmatrix} \frac{4}{11} & -\frac{3}{11} \\ \frac{1}{11} & \frac{2}{11} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} \frac{4}{11} & -\frac{3}{11} \\ \frac{1}{11} & \frac{2}{11} \end{bmatrix} \begin{bmatrix} 9 \\ 10 \end{bmatrix},$$

$$\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} \frac{6}{11} \\ \frac{29}{11} \end{bmatrix}, \quad \text{or} \quad x = \frac{6}{11}, \quad z = \frac{29}{11}.$$

(b) $$\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix},$$

$$\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} \\ \frac{3}{5} \end{bmatrix},$$

or $x = -\frac{1}{5}$, $z = \frac{3}{5}$.

(c) $$\begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} \frac{4}{11} & -\frac{3}{11} \\ \frac{1}{11} & \frac{2}{11} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so $y = w = 0$.

(d) $$\begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \end{bmatrix},$$

so $y = 1/5$, $w = 2/5$.

Parts (c) and (d) should be compared with parts (a) and (b), respectively. Since the matrices of coefficients are equal, the inverses do not have to be recomputed.

2-5. The Determinant Function

Many teachers have had experience with determinants in finding the solution of a system of linear equations. Seldom has it been pointed out to them that the determinant value is in a functional relationship with the coefficients of the variables. Although much time is spent in this section in developing theorems involving the determinant, the single most important idea is the assignment, or mapping,

$$\delta(X) : X \longrightarrow x \quad \text{for} \quad X \in M \quad \text{and} \quad x \in R.$$

Although this text does not dwell on determinants of matrices of higher order, any class that has had experience with determinants in advanced algebra can be shown the functional relationship that exists, through the determinant function, between 3×3 arrays of real coefficients on the one hand and the real numbers on the other.

It is important to recall the definition of a function. If with each element of a set A there is associated in some way exactly one element of a given set B, then this association constitutes a function from A to B. The essential point here is that a function pairs one and only one element of B with each element of A. The symbolism

[pages 76-83]

$$\delta : X \longrightarrow \delta(X)$$

is only suggestive, since nothing is said about the nature of the association. This association is defined by the open sentence: If

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$\delta(X) = ad - bc.$$

In order to specify any function completely, it is necessary to designate the domain of the independent variable and the range of the dependent variable. The domain of the determinant function is the complete set of 2×2 matrices. The range is the set of all real numbers. With each member A of the set of 2×2 matrices, there is associated a unique real number $r = ad - bc$. It is important to notice that this mapping gives a unique image, since this is an important criterion of a function. Since all images under the mapping are real numbers, we can perform all the usual operations on them. Through these operations many interesting properties of the determinant function can be demonstrated.

Exercises 2-5

1. (a) $\delta(A) = \delta \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = 5,$

$\delta(B) = \delta \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} = 2,$

$\delta(AB) = \delta \begin{bmatrix} 8 & 5 \\ 22 & 15 \end{bmatrix} = 120 - 110 = 10 = (5)(2) = \delta(A)\ \delta(B).$

(b) $\delta(A) = \delta \begin{bmatrix} t^2 & 1 \\ -1 & t \end{bmatrix} = t^3 + 1,$

$\delta(B) = \delta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1,$

$\delta(AB) = \delta \begin{bmatrix} 1 & t^2 \\ t & -1 \end{bmatrix} = -1 - t^3 = (t^3 + 1)(-1) = \delta(A)\ \delta(B).$

[pages 77-83]

(c) $\delta(A) = \delta\begin{bmatrix} x & x^2 \\ x^3 & x^4 \end{bmatrix} = 0,$

$$\delta(B) = \delta\begin{bmatrix} x & -x \\ 3 & 4 \end{bmatrix} = 7x,$$

$$\delta(AB) = \delta\begin{bmatrix} 4x^2 & 3x^2 \\ 4x^4 & 3x^4 \end{bmatrix} = 0 = (0)(7x) = \delta(A)\ \delta(B).$$

2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; then $tA = \begin{bmatrix} ta & tb \\ tc & td \end{bmatrix}$,

$$\delta(tA) = (ta)(td) - (tb)(tc) = t^2(ad - bc) = t^2\delta(A).$$

3.
$$\delta(A - tI) = \delta\begin{bmatrix} a - t & b \\ c & d - t \end{bmatrix} = (a - t)(d - t) - bc$$
$$= t^2 - t(a + d)t + ad - bc,$$

where

$$ad - bc = \text{constant term} = \delta(A).$$

4.
$$\delta(A) = \delta\begin{bmatrix} x & 1 \\ x^2 & -1 \end{bmatrix} = -x - x^2,$$

$$\delta(BAB^{-1}) = \delta\left(\begin{bmatrix} 2 & 1 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x & 1 \\ x^2 & -1 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}\right)$$

$$= \delta\left(\begin{bmatrix} 2x + x^2 & 1 \\ -5x - 2x^2 & -3 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}\right)$$

$$= \delta\begin{bmatrix} -4x - 2x^2 + 5 & -2x - x^2 + 2 \\ 10x + 4x^2 - 15 & 5x + 2x^2 - 6 \end{bmatrix}$$

$$= (-2x^2 - 4x + 5)(2x^2 + 5x - 6) - (-x^2 - 2x + 2)(4x^2 + 10x - 15)$$

$$= (-4x^4 - 18x^3 + 2x^2 + 49x - 30) - (-4x^4 - 18x^3 + 3x^2 + 50x - 30)$$

$$= -x^2 - x.$$

5. By Theorems 2–6 and 2–7, $\delta(BAB^{-1}) = \delta(B)\ \delta(AB^{-1}) = \delta(B)\ \delta(A)\ \delta(B^{-1})$

$$= \delta(B)\ \delta(A)\ \frac{1}{\delta(B)} = \delta(A).$$

[pages 83, 84]

6. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; then $A^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$,

$$\delta(A) = ad - bc, \quad \delta(A^t) = ad - cb = \delta(A), \quad \delta(AA^t) = \delta(A)\,\delta(A^t)$$
$$= (ad - bc)^2 \geq 0.$$

7. (a) $\delta \begin{bmatrix} 1 - t & 2 \\ 0 & 4 - t \end{bmatrix} = (1 - t)(4 - t)$; zeros are $1, 4$.

(b) $\delta \begin{bmatrix} -1 - t & 0 \\ 0 & 1 - t \end{bmatrix} = (-1 - t)(1 - t)$; zeros are $-1, 1$.

(c) $\delta \begin{bmatrix} 0 & 0 \\ -t & 1 - t \end{bmatrix} = 0$; zero for all t.

(d) $\delta \begin{bmatrix} a - t & 0 \\ 0 & b - t \end{bmatrix} = (a - t)(b - t)$; zeros are a, b.

8. $AA^t = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}$,

$$\delta(AA^t - xI) = \delta \begin{bmatrix} 4 - x & -2 \\ -2 & 2 - x \end{bmatrix} = (4 - x)(2 - x) - 4 = x^2 - 6x + 4;$$

$$A^tA = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix},$$

$$\delta(A^tA - xI) = \delta \begin{bmatrix} 5 - x & -1 \\ -1 & 1 - x \end{bmatrix} = (5 - x)(1 - x) - 1 = x^2 - 6x + 4$$
$$= \delta(AA^t - xI).$$

In general, a computation gives

$$\delta(AA^t - xI) = x^2 - x(a^2 + b^2 + c^2 + d^2) + (ad - bc)^2 = \delta(A^tA - xI).$$

2–6. The Group of Invertible Matrices

In this particular section, the mathematical concept of a group is introduced in a natural way. So far in this chapter, most of the discussion has centered around the operation of multiplication and the existence of an inverse. There should be no break now in the subject matter, nor should there be any abrupt digression in the point of view. The group concept evolves as a notion that binds the new ideas together. The more general definition of a group embraces the particular one that is stated first in terms of matrices. It will be an

[pages 84–92]

easy step later to proceed from the more specific to the general.

The illustrations and exercises are easily handled in terms of the specific definition of a group of matrices. Before introducing the general definition of a group, it would be well to give a considerable amount of collateral reading in order that the students will have some indication of the power and scope of the group concept.

Although the text gives several illustrations of both finite and infinite groups, there are a great many more that can readily be found and used for class illustration. If the students have had experience with the concept of congruence, they will find that this notion yields many simple examples. For instance,

$$\{1, 2, 4\} \bmod 7,$$
$$\{1, 3, 9\} \bmod 13,$$

and

$$\{1,3,5,7\} \bmod 8,$$

are groups under the operation of multiplication.

The exercises at the end of this section depend more on mathematical insight than those at the end of many other sections in the book, particularly those that have occurred up to this point. The exercises in the present section do not require much computational skill. If the class is not able to handle the exercises independently, do not despair!

Exercises 2–6

1. (a) No; not closed under multiplication.

   (b) Yes; all four properties can be read off from this multiplication table:

	I	−I	K	−K
I	I	−I	K	−K
−I	−I	I	−K	K
K	K	−K	I	−I
−K	−K	K	−I	I

2. The result follows quickly from the observation that I is a member of the

set, that the associativity law holds generally for 2×2 matrices, and that if

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \quad B = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$$

are elements of the set, then

$$AB = \begin{bmatrix} ab & 0 \\ 0 & ab \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{bmatrix}.$$

3. $$\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} c & d \\ d & c \end{bmatrix} = \begin{bmatrix} ac + bd & ad + bc \\ ad + bc & ac + bd \end{bmatrix},$$

and $\delta(AB) = \delta(A)\,\delta(B) = (1)(1) = 1$; thus if A and B are of the prescribed form, then so is AB, and accordingly the set is closed under multiplication. The associativity law holds as in Exercise 2, and I is an element of the set. Finally,

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ -b & a \end{bmatrix},$$

since $h = a^2 - b^2 = 1$, so that the inverse of any element of the set is a member of the set. Hence the set is a group.

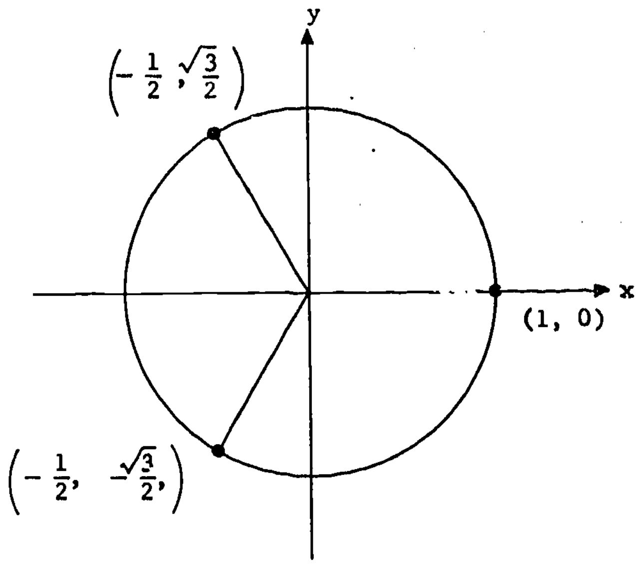
4. $$A^2 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}^2 = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix},$$

$$A^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

The group properties all follow from the multiplication table:

X	I	A	A^2
I	I	A	A^2
A	A	A^2	I
A^2	A^2	I	A

Note that $A^{-1} = A^2$, $(A^2)^{-1} = A$.



5. $TIT^{-1} = I$, $T(-I)T^{-1} = -(TIT^{-1}) = -I$, $K^2 = I$,

$$T = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}, \qquad K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$TK = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}, \qquad TKT^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}, \qquad T(-K)T^{-1} = \begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix}.$$

The multiplication table yields the four group properties:

X	$TIT^{-1} = I$	$T(-I)T^{-1} = -I$	TKT^{-1}	$T(-K)T^{-1}$
I	I	$-I$	TKT^{-1}	$T(-K)T^{-1}$
$-I$	$-I$	I	$T(-K)T^{-1}$	TKT^{-1}
TKT^{-1}	TKT^{-1}	$T(-K)T^{-1}$	I	$-I$
$T(-K)T^{-1}$	$T(-K)T^{-1}$	TKT^{-1}	$-I$	I

There is no restriction on T other than that T^{-1} exists. Hence, the table shows that we have a group provided T is invertible.

6. $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} ac & 0 \\ 0 & bd \end{bmatrix},$

and $(ac)(bd) = (ab)(cd) = (1)(1) = 1$; hence the product is of the desired form.

If $ab = 1$, then $a \neq 0$, $b \neq 0$, and therefore $1/a$ and $1/b$ exist. Hence

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix},$$

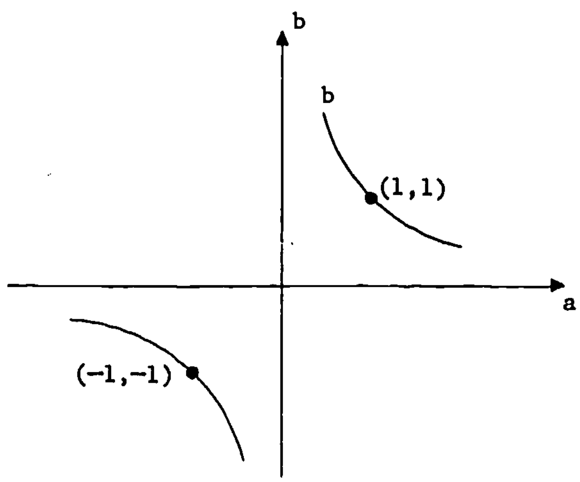
so that each element has an inverse; and

$$\left(\frac{1}{a}\right)\left(\frac{1}{b}\right) = \frac{1}{ab} = 1,$$

so that the inverse is of the desired form.

The identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a member, and the associative law holds as usual.

The graph of $\{(a,b)\colon ab = 1\}$ is a hyperbola:



7. Let

$$A = aI + bK = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \in H,$$

$$B = cI + dK = \begin{bmatrix} c & d \\ d & c \end{bmatrix} \in H.$$

(a) See Exercise 3.

(b) $\begin{bmatrix} x & y \\ y & x \end{bmatrix}^{-1} = \begin{bmatrix} \frac{x}{h} & -\frac{y}{h} \\ -\frac{y}{h} & \frac{x}{h} \end{bmatrix}$ provided $h = x^2 - y^2 \neq 0$.

(c) See Exercise 3.

8. (a) Let

$$G^t = \{A^t: A \in G\}, \quad A^t \in G^t, \quad B^t \in G^t.$$

Then

$$A^t B^t = (BA)^t.$$

Since $BA \in G$, it follows that $(BA)^t \in G^t$, so that G^t is closed under multiplication.

As we are dealing with 2×2 matrices, associativity holds.

G^t has an identity element for multiplication, since

$$I = I^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G^t.$$

We have already proved that $(A^t)^{-1} = (A^{-1})^t$. (See Exercise 2-4-4.) Since $A^{-1} \in G$, it follows that $(A^{-1})^t = (A^t)^{-1} \in G^t$.

Hence G^t is a group.

(b) Let $G' = \{BAB^{-1}: A \in G,\ B \in M$ with B fixed and invertible$\}$,

$$C' = BCB^{-1} \in G', \quad D' = BDB^{-1} \in G'.$$

Then

$$C'D' = (BCB^{-1})(BDB^{-1}) = BC(B^{-1}B)DB^{-1} = B(CD)B^{-1}.$$

Since $CD \in G$, it follows that $B(CD)B^{-1} \in G'$. Hence, $C'D' \in G'$ and closure is established.

Since we are dealing with 2×2 matrices, associativity holds.

Since $I' = BIB^{-1} = BB^{-1} = I$, it follows that $I' = I \in G'$, and G' has an identity element for multiplication.

Let $C' \in G'$ and consider

$$(C')^{-1} = (BCB^{-1})^{-1} = BC^{-1}B^{-1}.$$

Since $C \in G$, it follows that

$$C^{-1} \in G \text{ and } BC^{-1}B^{-1} \in G'.$$

But

$$C'(C')^{-1} = (BCB^{-1})(BC^{-1}B^{-1}) = BC(B^{-1}B)C^{-1}B^{-1} = B(CC^{-1})B^{-1} = BB^{-1} = I = I'.$$

Similarly, $(C')^{-1} C' = I$.

Hence G' is a group.

(a) Let $B \in G$. Then $B^{-1} \in G$, since G is a group. Accordingly,

$$B \in \{A^{-1}: A \in G\},$$

namely, B is the member obtained by setting $A = B^{-1}$. Conversely, let

$$B \in \{A^{-1}: A \in G\}.$$

Then $B = A^{-1}$ for some $A \in G$. But then $A^{-1} \in G$ since G is a group, and therefore $B = A^{-1} \in G$.

(b) Let $C \in G$. Then $B^{-1} C \in G$, and accordingly

$$C \in \{BA: A \in G\},$$

namely, C is the member obtained by setting $A = B^{-1} C$. Conversely, let

$$C \in \{BA: A \in G\}.$$

Then $C = BA$ for some $A \in G$. But also $B \in G$. Therefore, $C = BA \in G$ since G is a group.

(a) The set of odd integers does not form a group under addition, since it is not closed under this operation. For example, $3 + 5 = 8$, and 8 is not odd.

(b) The set R^+ of positive real numbers does form a group under multiplication.

If $a \in R^+$, $b \in R^+$, then $ab \in R^+$, so we have closure under multiplication.

If $a \in R^+$, $b \in R^+$, $c \in R^+$, then $(ab)c = a(bc)$, and we have associativity under multiplication.

$1 \in R^+$, so there is a multiplicative identity.

If $a \in R^+$, then $\frac{1}{a} \in R^+$ and $(a)\left(\frac{1}{a}\right) = \left(\frac{1}{a}\right)(a) = 1$. Hence, each number of R^+ has a multiplicative inverse.

(c) $A = \{1, -1, i, -i\}$ is a group under multiplication. We shall examine the multiplication table:

	1	−1	i	−i
1	1	−1	i	−i
−1	−1	1	−i	i
i	i	−i	−1	1
−i	−i	i	1	−1

The body of the table contains only members of the original set A, and so A is closed under multiplication.

Complex numbers are associative under multiplication. We could also verify this by checking all possible products of three factors: $(ab)c = a(bc)$.

The element 1 serves as an identity element.

Each element of A has an inverse element in A, as can be found from the table. Thus

$$1^{-1} = 1,$$
$$(-1)^{-1} = -1,$$
$$(i)^{-1} = -i,$$
$$(-i)^{-1} = i.$$

(d) Let $T = \{3m: m \text{ is an integer}\}$. Let a, b, and c be arbitrary integers, so that 3a, 3b, and 3c are arbitrary members of T. Then

[page 94]

$$3a + 3b = 3(a + b),$$

$$(3a + 3b) + (3c) = (3a) + (3b + 3c),$$

so that the closure and associativity properties hold.

We have $3a + 0 = 3a$, for $3a \in T$; and $(3)(0) = 0 \in T$. Hence there is an additive inverse in T.

We have $3a + (-3a) = 0$, and $-3a = 3(-a) \in T$, since $-a$ is an integer. Hence every element $3a \in T$ has the additive inverse $-3a$ also in T.

Thus T is a group under addition.

11. If

$$a \circ b = a \circ c,$$

then

$$a^{-1} \circ (a \circ b) = a^{-1} \circ (a \circ c),$$

$$(a^{-1} \circ a) \circ b = (a^{-1} \circ a) \circ c,$$

$$i \circ b = i \circ c,$$

$$b = c.$$

2-7. An Isomorphism between Complex Numbers and Matrices

In this section, as in its predecessor, powerful mathematical ideas are introduced in an easy manner. The new ideas seem to arise from the context. In both sections, a rich background through which to make associations is most helpful, though certainly not necessary. The text itself introduces a considerable amount of rich material that is easy to handle. For the class that has not been using complex numbers recently, a short review of these numbers and their operations is in order.

When the class truly understands how it is that the algebra of complex numbers is embedded in the algebra of matrices, a very significant goal will have been attained.

[pages 94-101]

Exercises 2-7

1. (a) $(1 - i) + (0 - 2i) \qquad (I - J) + (\underline{0} - 2J) =$

$= 1 - 3i \quad \longleftrightarrow \quad I - 3J,$

and

$(1 - i)(0 - 2i) \qquad (I - J)(\underline{0} - 2J) =$

$= - 2i + 2i^2 \qquad - 2IJ + 2J^2 =$

$\qquad - 2J - 2I =$

$= - 2 - 2i \quad \longleftrightarrow \quad - 2I - 2J.$

(b) $(3 - 4i) + (1 + i) \qquad (3I - 4J) + (I + J) =$

$= 4 - 3i \quad \longleftrightarrow \quad 4I - 3J,$

and

$(3 - 4i)(1 + i) \qquad (3I - 4J)(I + J) =$

$= 3(1) + 3i - 4i - 4i^2 \qquad 3I + 3J - 4J - 4J^2 =$

$= 7 - i \quad \longleftrightarrow \quad 7I - J.$

(c) $(0 - 5i) + (3 + 4i) \qquad (0 - 5J) + (3I + 4J) =$

$= 3 - i \quad \longleftrightarrow \quad 3I - J,$

and

$(0 - 5i)(3 + 4i) \qquad (0 - 5J)(3I + 4J) =$

$= 20 - 15i \quad \longleftrightarrow \quad 20I - 15J.$

2. $$x_1 \longleftrightarrow \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix},$$

$$x_2 \longleftrightarrow \begin{bmatrix} x_2 & 0 \\ 0 & x_2 \end{bmatrix};$$

$$x_1 + x_2 \qquad \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix} + \begin{bmatrix} x_2 & 0 \\ 0 & x_2 \end{bmatrix} =$$

$$= (x_1 + x_2) \longleftrightarrow \begin{bmatrix} x_1 + x_2 & 0 \\ 0 & x_1 + x_2 \end{bmatrix};$$

$$x_1x_2 \longleftrightarrow \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix}\begin{bmatrix} x_2 & 0 \\ 0 & x_2 \end{bmatrix} =$$

$$= (x_1x_2) \longleftrightarrow \begin{bmatrix} x_1x_2 & 0 \\ 0 & x_1x_2 \end{bmatrix};$$

$$0 \quad \longleftrightarrow \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix};$$

$$1 \quad \longleftrightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

$$-x \quad \longleftrightarrow \quad \begin{bmatrix} -x & 0 \\ 0 & -x \end{bmatrix};$$

$$\frac{1}{x} \quad \longleftrightarrow \quad \begin{bmatrix} \frac{1}{x} & 0 \\ 0 & \frac{1}{x} \end{bmatrix} \quad \text{for } x \neq 0.$$

3. (a) True: $f(x + y) = \begin{bmatrix} x + y & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix} = f(x) + f(y).$

(b) True: $f(xy) = \begin{bmatrix} xy & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix} = f(x)\, f(y).$

(c) True: $f(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \underline{0}.$

(d) False: $f(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

(e) False: $f(x) = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$

has no inverse.

Relative to parts (d) and (e), however, within the class of matrices $f(x)$ the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = f(1)$ plays the role of unit element in that

$$f(x)\, f(1) = f(x) = f(1)\, f(x), \quad x \in R;$$

and the matrix

$$\begin{bmatrix} \frac{1}{x} & 0 \\ 0 & 0 \end{bmatrix} = f\left(\frac{1}{x}\right), \quad x \in R, \quad x \neq 0,$$

plays the role of the reciprocal (or multiplicative inverse) of $f(x)$ in that

$$f(x)\, f\left(\frac{1}{x}\right) = f(1) = f\left(\frac{1}{x}\right)\, f(x).$$

4. Let $x \in G$ and $y \in G$:

$$x = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \quad y = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}, \quad \delta(x) = \delta(y) = 1.$$

Then

$$xy = \begin{bmatrix} ac - bd & ad + bc \\ -ad - bc & ac - bd \end{bmatrix};$$

and since

$$\delta(xy) = \delta(x)\,\delta(y) = (1)(1) = 1,$$

it follows that $xy \in G$. Hence, G is closed under multiplication.

Also, G is associative under multiplication, since 2×2 matrices have this property in general.

Next,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G,$$

since $\delta(I) = 1$ and 1 is a rational number.

If $x \in G$, then

$$x^{-1} = \begin{bmatrix} \frac{a}{h} & -\frac{b}{h} \\ \frac{b}{h} & \frac{a}{h} \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

since $h = a^2 + b^2 = 1$. Hence, G has the inverse property for multiplication.

Accordingly, G is a group.

2-8. Algebras

This section is a summary of Chapter 2. As is the case with all summaries, it is superfluous if the work of the chapter has brought the class to the proper degree of mathematical maturity. By now, the student will clearly understand that there are many algebras —— different, but not entirely different, from the algebra of real numbers. He will begin to understand the scope and imaginative qualities of the discipline and to realize that the student of mathematics has as much room to express himself as the student in any other discipline.

Chapter 3

3-1. Equivalent Systems

As mentioned earlier in this Commentary for Teachers, Chapter 3 may be undertaken immediately following Chapter I. Chapter 2, which is concerned with the algebra of 2×2 matrices, constitutes a unit by itself, and the chapter is not a prerequisite for Chapter 3. The operations introduced in Chapter 1 are sufficient for the material in Chapter 3. Undoubtedly there will be many teachers who feel that it is better to cover Chapter 3 before Chapter 2 in order to solidify the students' mastery of the operations defined in Chapter 1.

The over-all purpose of this chapter is to introduce the use of matrices as a means of solving linear systems, to familiarize the student with the inverse of a matrix and with a method for finding the entries for the inverse, and to show how to use the inverse in practical situations.

In beginning the chapter, the teacher will probably wish to review briefly the usual methods for solving systems of linear equations as introduced in the students' earlier algebra courses. It may be well to start with systems of two variables and to consider the methods of addition and subtraction, substitution, and graphing.

We have placed a great deal of emphasis on the notion of equivalent systems in this section, and graphic solutions of two-variable systems offer an excellent opportunity to display this equivalence visually.

For instance, consider the system:

$$2x - y = 3, \qquad (1)$$

$$-5x + 3y = -7. \qquad (2)$$

A solution by addition leads next to the system

$$2x - y = 3, \qquad (3)$$

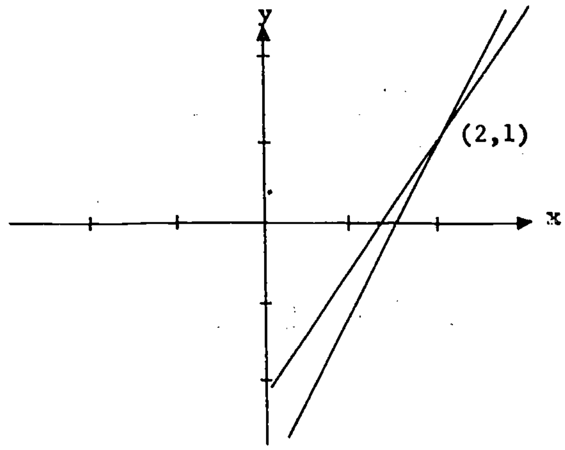
$$x = 2, \qquad (4)$$

followed by

$$x = 2, \qquad (5)$$

$$y = 1. \qquad (6)$$

We may represent (1) and (2) graphically by

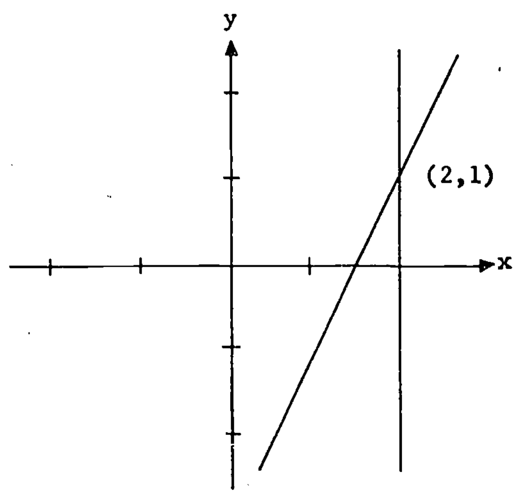


From this, the solution $(x,y) = (2,1)$ can easily be observed.

Now the system of equations

$$2x - y = 3, \quad x = 2,$$

which was obtained by algebraic operations from the first system, can also be graphed, thus:



From this graph, it is quickly noted that the solution is again

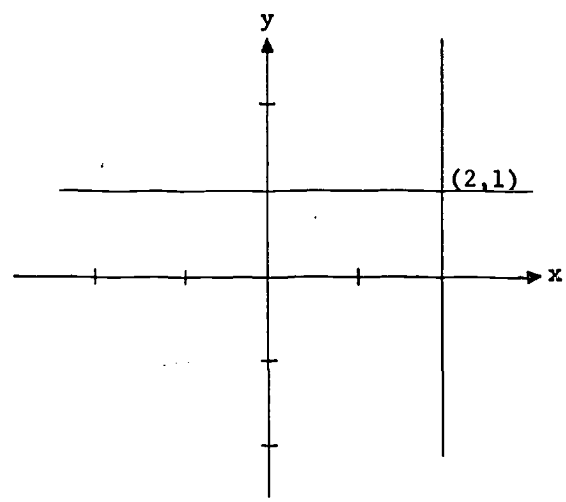
[pages 103-106]

$(x,y) = (2,1)$.

If we continue our algebraic operations, we can finally obtain the system

$$x = 2, \quad y = 1.$$

Again we have two equations that can be represented graphically:



From this graph, it is immediately seen that the solution is once again $(x,y) = (2,1)$.

From these graphs, the student can readily appreciate the fact that all three systems have the same solution. Since equivalent systems are systems having the same solution, it follows by definition that the three systems designated above are equivalent. The graphs are not a proof of this, but they are aids in understanding the operations and their impact on the systems.

Solutions of two-variable systems can be extended to apply to three-variable systems; it is unlikely, however, that the student will ever have developed a solution in exactly the manner of the example in the text. In class demonstrations, the teacher should always use the pattern of the text and emphasize the systematic nature, pointing out that this technique, while different in structure, is very similar to the usual addition method of solution. The procedure used here, however, is one that generalizes easily and that lays a foundation for the matrix methods you will subsequently employ. The emphasis throughout this section should be on equivalence.

[pages 103-106]

The earlier exercises provide practice in using the technique of solution developed in this section. The last exercises are provided to lead into later work.

Exercises 3-1

1\. (a) $3x + 4y = 4,\quad 5x + 7y = 1;$

$$\begin{aligned} 15x + 20y &= 20, \\ 15x + 21y &= 3, \\ -y &= 17, \\ y &= -17; \end{aligned} \qquad \begin{aligned} 21x + 28y &= 28, \\ 20x + 28y &= 4, \\ x &= 24. \end{aligned}$$

(b) $x - 2y = 3,\quad y = 2;$

$$\begin{aligned} x - 2y &= 3, \\ 2y &= 4, \\ x &= 7. \end{aligned}$$

(c) $x + y - z = 3,\quad 2y + z = 10,\quad 5x - y - 2z = -3;$

$$\begin{aligned} x + 3y &= 13, \\ 2y + z &= 10, \\ 5x + 3y &= 17; \end{aligned} \qquad 4x = 4,\quad 3y = 12, \quad x = 1,\quad y = 4,\quad z = 2.$$

(d) $x - 3y + 2z = 6,\quad y - z = -4,\quad z = 7,\qquad y = 3,\qquad x = 1.$

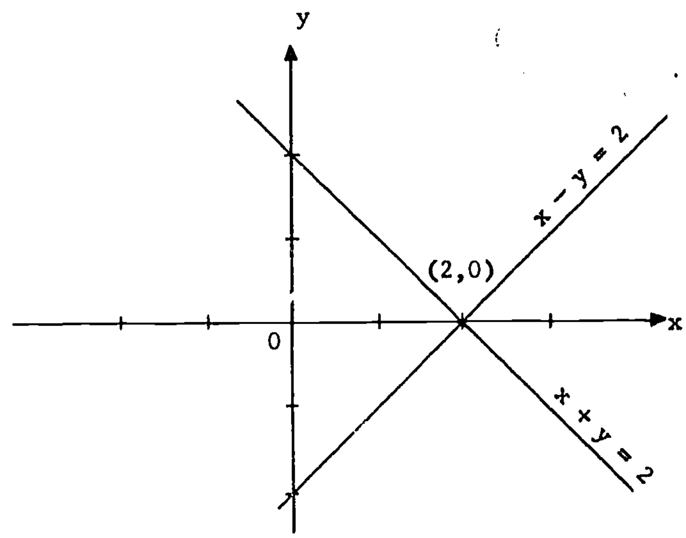
(e) $x + 2y + z - 3w = 2,\quad y - 2z + w = 7,\quad z - 2w = 0,\quad w = 3,\qquad z = 6,\qquad y = 16,\qquad x = -27.$

(f) $x = a,\quad y = b,\quad z = c,\quad w = d.$

[pages 103-106]

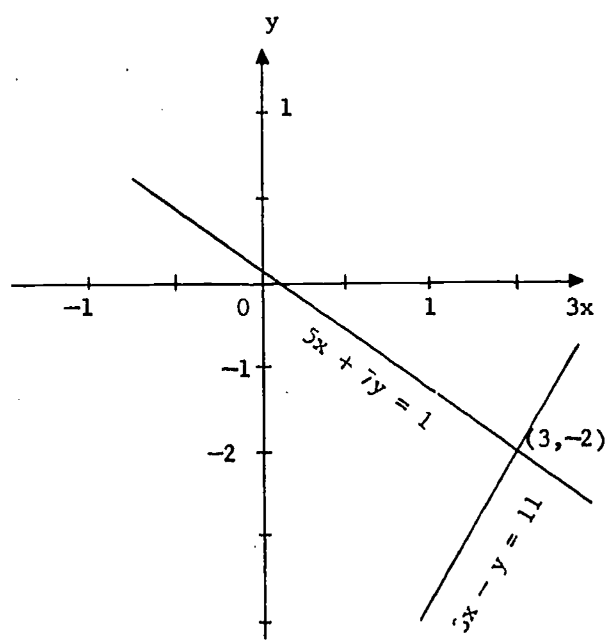
2. (a) $x + y = 2,$
$x - y = 2;$

$x = 2,$
$y = 0.$



(b) $3x - y = 11,$
$5x + 7y = 1;$

$x = 3,$
$y = -2.$



3. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & x \\ b & y \\ c & z \end{bmatrix} = \begin{bmatrix} a & x \\ b & y \\ c & z \end{bmatrix}.$

4.
$$\begin{array}{l} x = 2, \\ y = 3, \\ z = -1, \end{array} \longrightarrow \begin{array}{l} x + 0y + 0z = 2, \\ 0x + y + 0z = 3, \\ 0x + 0y + z = -1, \end{array} \longrightarrow \begin{array}{l} 2x + 0y + 0z = 4, \\ 0x + y + 0z = 3, \\ 0x + 0y + z = -1, \end{array}$$

$$\longrightarrow \begin{array}{l} 2x - 3y + 0z = -5, \\ 0x + y + 0z = 3, \\ 0x + 0y + z = -1, \end{array} \longrightarrow \begin{array}{l} 2x - 3y + z = -6, \\ 0x + 2y + 0z = 6, \\ 0x + 0y + 3z = -3, \end{array} \longrightarrow \begin{array}{l} 2x - 3y + z = -6, \\ x + 2y + 0z = 8, \\ 0x + 0y + 3z = -3. \end{array}$$

$$\longrightarrow \begin{array}{l} 2x - 3y + z = -6, \\ x + 2y - z = 9, \\ 0x + 0y + 3z = -3, \end{array} \longrightarrow \begin{array}{l} 2x - 3y + z = -6, \\ x + 2y - z = 9, \\ 3x + 0y + 3z = 3, \end{array} \longrightarrow \begin{array}{l} 2x - 3y + z = -6, \\ x + 2y - z = 9, \\ 3x + y + 3z = 6. \end{array}$$

Since the solutions of A are solutions of B, the two systems are equivalent.

5. (a)
$$\begin{array}{l} x + 2y - z = 3, \\ x - y + z = 4, \\ 4x - y + 2z = 14; \end{array}$$
$$\begin{array}{l} 2x + y = 7, \\ 2x + y = 6. \end{array}$$

Since there exist no values of x and y that satisfy the two equations thus obtained, the solution set is $\emptyset$.

(b)
$$\begin{array}{l} x + 2y - z = 3, \\ x - y + z = 4, \\ 4x - y + 2z = 15. \end{array}$$
$$\begin{array}{l} 2x + y = 7, \\ 2x + y = 7. \end{array}$$

Therefore an infinite number of values of x and y satisfy the two equations obtained.

For $x = 0$, $y = 7$, we get $z = 11$;
$x = 3$, $y = 1$, we get $z = 2$;
$x = 1$, $y = 5$, we get $z = 8$.

3-2. Formulation in Terms of Matrices

There are two major ideas introduced in this section. The first centers around our ability to represent systems of linear equations in matrix form. This

[pages 106-112]

is an enormous accomplishment. The quiet statement on page 108–109:

> "It is an achievement not to be taken modestly that we are able to consider and work with a large system of equations in terms of such a simple representation as $AX = B$."

is one of the most significant in the book. The concept of the matrix equation $AX = B$ leads naturally to the second of the important ideas in this section, that of the matrix function. If chapter two was covered, the student is familiar with the determinant function

$$\delta : M \longrightarrow R,$$

having a set of matrices as domain and as range a set of real numbers. In this chapter, we introduce a new function

$$f : X \longrightarrow Y,$$

where both the domain and range are sets of matrices. Although we are concerned only with the problem of finding the matrix in the domain that maps onto a specified matrix in the range, the concept of a function from matrices onto matrices is used in later chapters. Since it is one that arises naturally from matrix equations, we include it at this time.

The principal <u>technique</u> to be gained in this section is simply that of expressing linear systems in matrix form. We shall put it to use in Sections 3.4 and 3.5.

The exercises provide work in both the techniques and the underlying ideas developed in this section. In particular, Exercise 3 should be worked out in the form used in Section 3–1 in order to familiarize the student with a pattern to be duplicated later in matrix form.

<u>Exercise 3–2</u>

1. (a) $$\begin{bmatrix} 4 & -2 & 7 \\ 3 & 1 & 5 \\ 0 & 6 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

   (b) $$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

2. (a)
$$\begin{aligned} 3x + 4y + 5z &= 1,\\ x + 2y + 3z &= 0,\\ y + 2z &= 2. \end{aligned}$$

(b)
$$\begin{aligned} 3x + 2y - 2z &= 1,\\ 2x - y - 4z &= 2,\\ -x + y + 5z &= 3,\\ 3u + 2v - 2w &= 2,\\ 2u - v - 4w &= 3,\\ -u + v + 5w &= 1. \end{aligned}$$

3.
$$\begin{aligned} x + y + z - w &= 1,\\ x - y + 3z + 2w &= 2,\\ 2x + y + 3z + w &= -2,\\ x - 2y + z + 3w &= 10, \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 1 & -1\\ 1 & -1 & 3 & 2\\ 2 & 1 & 3 & 1\\ 1 & -2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x\\ y\\ z\\ w \end{bmatrix} = \begin{bmatrix} 1\\ 2\\ -2\\ 10 \end{bmatrix};$$

$$\begin{aligned} x + y + z - w &= 1,\\ 2x + 0 + 4z + w &= 3,\\ 3x + 0 + 6z + 3w &= 0,\\ 3x + 0 + 3z + w &= 12, \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 1 & -1\\ 2 & 0 & 4 & 1\\ 3 & 0 & 6 & 3\\ 3 & 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} x\\ y\\ z\\ w \end{bmatrix} = \begin{bmatrix} 1\\ 3\\ 0\\ 12 \end{bmatrix};$$

$$\begin{aligned} x + y + z - w &= 1,\\ 0 + 0 + 0 - w &= 3,\\ 0 + 0 + 3z + 2w &= -12,\\ 3x + 0 + 3z + w &= 12, \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 1 & -1\\ 0 & 0 & 0 & -1\\ 0 & 0 & 3 & 2\\ 3 & 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} x\\ y\\ z\\ w \end{bmatrix} = \begin{bmatrix} 1\\ 3\\ -12\\ 12 \end{bmatrix};$$

$$\begin{aligned} x + y + z + 0 &= -2,\\ 0 + 0 + 0 - w &= 3,\\ 0 + 0 + 3z + 0 &= -6,\\ 3x + 0 + 3z + w &= 12, \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 1 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 3 & 0\\ 3 & 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} x\\ y\\ z\\ w \end{bmatrix} = \begin{bmatrix} -2\\ 3\\ -6\\ 12 \end{bmatrix};$$

$$\begin{aligned} x + y + 0 + 0 &= 0,\\ 0 + 0 + 0 - w &= 3,\\ 0 + 0 + z + 0 &= -2,\\ 3x + 0 + 0 + w &= 18, \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\\ 3 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x\\ y\\ z\\ w \end{bmatrix} = \begin{bmatrix} 0\\ 3\\ -2\\ 18 \end{bmatrix};$$

[pages 112-113]

$$\begin{aligned} x + y + 0 + 0 &= 0, \\ 0 + 0 + 0 - w &= 3, \\ 0 + 0 + z + 0 &= -2, \\ x + 0 + 0 + 0 &= 7, \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -2 \\ 7 \end{bmatrix};$$

$$\begin{aligned} 0 + y + 0 + 0 &= -7, \\ 0 + 0 + 0 + w &= -3, \\ 0 + 0 + z + 0 &= -2, \\ x + 0 + 0 + 0 &= 7, \end{aligned} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ -2 \\ 7 \end{bmatrix};$$

$$\begin{aligned} x + 0 + 0 + 0 &= 7, \\ 0 + y + 0 + 0 &= -7, \\ 0 + 0 + z + 0 &= -2, \\ 0 + 0 + 0 + w &= -3, \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \\ -2 \\ -3 \end{bmatrix};$$

$$\begin{aligned} x &= 7, \\ y &= -7, \\ z &= -2, \\ w &= -3. \end{aligned}$$

4. (a) $Y = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \end{bmatrix}.$$

(b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$,

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix},$$

from which

$$y = \frac{7}{2} \quad \text{and} \quad x = -4.$$

5. $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}$, $\quad Y = \begin{bmatrix} y_1 \end{bmatrix}$, $\quad a_i, x_i, y_1 \in R.$

$$AX = Y,$$

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} X = \begin{bmatrix} y_1 \end{bmatrix},$$

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \end{bmatrix},$$

$$\begin{bmatrix} a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 \end{bmatrix} = \begin{bmatrix} y_1 \end{bmatrix}.$$

The domain of X is the set of matrices

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Since $AX = Y$ is not one-to-one from X to Y, there exists no inverse; for example, if $a_1 = a_2 = a_3 = a_4 = 1$, then

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

are both mapped onto $\begin{bmatrix} 1 \end{bmatrix}$.

3-3. Inverse of a Matrix

The principal ideas advanced in this section are those of row operations and row equivalence. Finally, we combine these ideas to produce the inverse of a 3×3 matrix. These concepts can be developed quite independently; they do not depend on linear systems in any way. In fact the common practice in more advanced texts is to introduce linear systems after these concepts have been developed. It is the purpose of the present text, however, to provide something concrete upon which to build the students' thinking; thus our development is interwoven with work on linear systems.

[pages 113-116]

The present test gives a method for determining the inverse of a square matrix, provided that it exists. It is definitely not the only method. We now present a popular alternative method of finding an inverse so that, if you wish, you may present it to the class and dispel any notion the students may have about the uniqueness of the method they are learning. First we must define a minor and a cofactor.

Let us consider the general 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

If we delete one row and one column, we have left a matrix consisting of two rows and two columns. If we delete the row that contains e as an entry and the column that contains e as an entry, we have a 2×2 matrix. The determinant of this matrix is called the <u>minor</u> of A corresponding to e.

$$\begin{bmatrix} a & \cancel{b} & c \\ \cancel{d} & \cancel{e} & \cancel{f} \\ g & \cancel{h} & i \end{bmatrix} \text{ becomes } \begin{bmatrix} a & c \\ g & i \end{bmatrix},$$

and $ai - cg$ is the minor for e. Similarly, $dh - eg$ is the minor for c:

$$\begin{bmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & e & \cancel{f} \\ g & h & \cancel{i} \end{bmatrix}.$$

Now to form the inverse A^{-1} of a 3×3 matrix A, we first write the matrix having as entries the minors corresponding to the respective entries of A. Hence we have

$$\begin{bmatrix} ei - fh & di - fg & dh - eg \\ bi - ch & ai - cg & ah - bg \\ bf - ce & af - cd & ae - bd \end{bmatrix}.$$

Next we write the transpose of this new matrix:

$$\begin{bmatrix} ei - fh & bi - ch & bf - ce \\ di - fg & ai - cg & af - cd \\ dh - eg & ah - bg & ae - bd \end{bmatrix}.$$

Finally the inverse is formed by alternating the signs of the terms (the minors with signs thus chosen are called cofactors) and then dividing each term by $\delta(A)$, where

$$\delta(A) = aei + bfg + chd - gec - hfa - ibd.$$

Thus

$$A^{-1} = \begin{bmatrix} \frac{ei - fh}{\delta(A)} & -\frac{bi - ch}{\delta(A)} & \frac{bf - ce}{\delta(A)} \\ -\frac{di - fg}{\delta(A)} & \frac{ai - cg}{\delta(A)} & -\frac{af - cd}{\delta(A)} \\ \frac{dh - eg}{\delta(A)} & -\frac{ah - bg}{\delta(A)} & \frac{ae - bd}{\delta(A)} \end{bmatrix}.$$

If you care to, you can develop this particular form by carrying through the proper series of row operations on A and I; but it is a herculean task. Have many sheets of paper and be prepared to spend a great deal of time.

As in the case of 2×2 matrices, if

$$\delta(A) = 0,$$

then A does not have an inverse.

The exercises provide practice in finding matrix inverses. The student should discern for himself through Exercises 3 to 8 some of the ideas developed in the following Section 3–4.

Exercises 3–3

1. (a) $\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$

$\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$

$\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix};$

$\begin{bmatrix} 1 & 5 \\ 0 & -7 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix};$

Check:

$$\begin{bmatrix} \frac{2}{7} & -\frac{3}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{2}{7} & -\frac{3}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}.$$

(b) $\begin{bmatrix} 0 & 3 \\ 4 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$

$$\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix};$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix};$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{4} \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{6} & \frac{1}{4} \\ \frac{1}{3} & 0 \end{bmatrix}.$$

Check:

$$\begin{bmatrix} \frac{1}{6} & \frac{1}{4} \\ \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(c) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix};$$

$$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix};$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{6} \\ 0 & 1 & 1 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 1 \\ \frac{1}{3} & 0 & \frac{1}{6} \end{bmatrix}.$$

Check:

$$\begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 1 \\ \frac{1}{3} & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(d) $$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & -\frac{1}{5} \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & -\frac{1}{5} \end{bmatrix}.$$

Check:

$$\begin{bmatrix} -\frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2. (a) $\begin{bmatrix} 2 & 2 \\ -1 & \sqrt{2} \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$;

$\begin{bmatrix} 1 & 1 \\ -1 & \sqrt{2} \end{bmatrix}$, $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$;

$\begin{bmatrix} 1 & 1 \\ 0 & 1 + \sqrt{2} \end{bmatrix}$, $\begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$;

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2(1 + \sqrt{2})} & \frac{1}{1 + \sqrt{2}} \end{bmatrix}$;

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} \frac{1}{2} - \frac{1}{2(1 + \sqrt{2})} & -\frac{1}{1 + \sqrt{2}} \\ \frac{1}{2(1 + \sqrt{2})} & \frac{1}{1 + \sqrt{2}} \end{bmatrix}$;

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} \frac{\sqrt{2}}{2(1 + \sqrt{2})} & -\frac{1}{1 + \sqrt{2}} \\ \frac{1}{2(1 + \sqrt{2})} & \frac{1}{1 + \sqrt{2}} \end{bmatrix}$.

Check:

$$\begin{bmatrix} \frac{\sqrt{2}}{2(1 + \sqrt{2})} & -\frac{1}{1 + \sqrt{2}} \\ \frac{1}{2(1 + \sqrt{2})} & \frac{1}{1 + \sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) $\begin{bmatrix} 4 & -2 \\ 6 & -3 \end{bmatrix}$, $\delta(A) = 0$, so no inverse.

(c) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 2 & 3 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$;

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ 2 & 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Since the left matrix will not reduce to the identity matrix, there exists no inverse.

(d) $$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 5 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 6 & 0 \\ 0 & 5 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 5 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 1 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix};$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{12} & -\frac{5}{12} & \frac{1}{12} \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{12} & -\frac{5}{12} & \frac{1}{12} \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{7}{12} & -\frac{1}{12} & \frac{5}{12} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{12} & -\frac{5}{12} & \frac{1}{12} \end{bmatrix}.$$

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Check:

$$\begin{bmatrix} -\frac{7}{12} & -\frac{1}{12} & \frac{5}{12} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{12} & -\frac{5}{12} & \frac{1}{12} \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(e) $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 3 & 6 \\ 5 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$

$\begin{bmatrix} 1 & 4 & 7 \\ 32 & 9 & 0 \\ 5 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix};$

$\begin{bmatrix} 36 & 11 & 0 \\ 32 & 9 & 0 \\ 5 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix};$

$\begin{bmatrix} 4 & 2 & 0 \\ 32 & 9 & 0 \\ 5 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix};$

$\begin{bmatrix} 4 & 2 & 0 \\ 0 & -7 & 0 \\ 5 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 \\ -8 & 9 & -2 \\ 0 & 0 & 1 \end{bmatrix};$

$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -5 & -1 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{8}{7} & -\frac{9}{7} & \frac{2}{7} \\ 0 & 0 & -1 \end{bmatrix};$

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & -1 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{9}{14} & \frac{11}{14} & \frac{3}{14} \\ \frac{8}{7} & -\frac{9}{7} & \frac{2}{7} \\ 0 & 0 & -1 \end{bmatrix};$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{9}{28} & \frac{11}{28} & \frac{3}{28} \\ \frac{8}{7} & -\frac{9}{7} & \frac{2}{7} \\ \frac{8}{7} & -\frac{9}{7} & -\frac{5}{7} \end{bmatrix};$

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{9}{28} & \frac{11}{28} & \frac{3}{28} \\ \frac{8}{7} & -\frac{9}{7} & \frac{2}{7} \\ -\frac{13}{28} & \frac{19}{28} & -\frac{5}{28} \end{bmatrix}.$$

Check:

$$\begin{bmatrix} -\frac{9}{28} & \frac{11}{28} & \frac{3}{28} \\ \frac{8}{7} & -\frac{9}{7} & \frac{2}{7} \\ -\frac{13}{28} & \frac{19}{28} & -\frac{5}{28} \end{bmatrix} \begin{bmatrix} 1 & 4 & 7 \\ 2 & 3 & 6 \\ 5 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. (a) $$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 3 & 6 \\ 5 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix},$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{9}{28} & \frac{11}{28} & \frac{3}{28} \\ \frac{8}{7} & -\frac{9}{7} & \frac{2}{7} \\ -\frac{13}{28} & \frac{19}{28} & -\frac{5}{28} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} -\frac{3}{14} \\ \frac{38}{7} \\ -\frac{37}{14} \end{bmatrix}.$$

(b) $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{9}{28} & \frac{11}{28} & \frac{3}{28} \\ \frac{8}{7} & -\frac{9}{7} & \frac{2}{7} \\ -\frac{13}{28} & \frac{19}{28} & -\frac{5}{28} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{33}{28} \\ \frac{20}{7} \\ \frac{29}{28} \end{bmatrix}.$$

(c) $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{9}{28} & \frac{11}{28} & \frac{3}{28} \\ \frac{8}{7} & -\frac{9}{7} & \frac{2}{7} \\ -\frac{13}{28} & \frac{19}{28} & -\frac{5}{28} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} \\ -\frac{27}{7} \\ \frac{16}{7} \end{bmatrix}.$$

(d) $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

4. $$\begin{bmatrix} x & u & m & r \\ y & v & n & s \\ z & w & p & t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{7}{8} & \frac{1}{8} & \frac{3}{8} \\ \frac{5}{8} & -\frac{3}{8} & -\frac{1}{8} \end{bmatrix} \begin{bmatrix} 3 & 1 & -6 & \frac{3}{2} \\ 0 & -1 & 1 & 0 \\ 4 & 3 & -9 & \frac{11}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -1 & 2 \\ -\frac{9}{8} & \frac{1}{8} & 2 & \frac{3}{4} \\ \frac{11}{8} & \frac{5}{8} & -3 & \frac{1}{4} \end{bmatrix}.$$

5. (a) $$\begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 12 & 10 \\ 6 & -13 & -8 \\ -1 & 5 & 7 \end{bmatrix} \frac{1}{17}$$

$$= \frac{1}{17} \begin{bmatrix} 17 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 17 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) $$\frac{1}{17} \begin{bmatrix} 1 & 12 & 10 \\ 6 & -13 & -8 \\ -1 & 5 & 7 \end{bmatrix} \begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix}$$

$$= \frac{1}{17} \begin{bmatrix} 17 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 17 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

6. $$\begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 12 & 10 \\ 6 & -13 & -8 \\ -1 & 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 17 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 17 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 51 \\ 34 \\ -17 \end{bmatrix},$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}.$$

7. $$\begin{aligned} 2x + y + 2z - 3w &= 0, \\ 4x + y + z + w &= 15, \\ 6x - y - z - w &= 5, \\ 4x - 2y + 3z - w &= 2; \end{aligned}$$

$$\begin{bmatrix} 2 & 1 & 2 & -3 \\ 4 & 1 & 1 & 1 \\ 6 & -1 & -1 & -1 \\ 4 & -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \\ 5 \\ 2 \end{bmatrix} ;$$

$$\begin{bmatrix} 2 & 1 & 2 & -3 \\ 4 & 1 & 1 & 1 \\ 6 & -1 & -1 & -1 \\ 4 & -2 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ;$$

$$\begin{bmatrix} 2 & 1 & 2 & -3 \\ 1 & 0 & 0 & 0 \\ 6 & -1 & -1 & -1 \\ 4 & -2 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{10} & \frac{1}{10} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ;$$

$$\begin{bmatrix} 0 & 1 & 2 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & -2 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -\frac{1}{5} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{10} & \frac{1}{10} & 0 \\ 0 & -\frac{3}{5} & \frac{2}{5} & 0 \\ 0 & -\frac{2}{5} & -\frac{2}{5} & 1 \end{bmatrix} ;$$

$$\begin{bmatrix} 0 & 0 & 1 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 5 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -\frac{4}{5} & \frac{1}{5} & 0 \\ 0 & \frac{1}{10} & \frac{1}{10} & 0 \\ 0 & -\frac{3}{5} & \frac{2}{5} & 0 \\ 0 & \frac{4}{5} & -\frac{6}{5} & 1 \end{bmatrix} ;$$

$$\begin{bmatrix} 0 & 0 & 1 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 5 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -\frac{4}{5} & \frac{1}{5} & 0 \\ 0 & \frac{1}{10} & \frac{1}{10} & 0 \\ -1 & \frac{7}{5} & -\frac{3}{5} & 0 \\ 0 & \frac{4}{5} & -\frac{6}{5} & 1 \end{bmatrix} ;$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 5 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{21} & \frac{4}{35} & -\frac{23}{105} & \frac{1}{21} \\ 0 & \frac{1}{10} & \frac{1}{10} & 0 \\ -1 & \frac{7}{5} & -\frac{3}{5} & 0 \\ 0 & \frac{4}{5} & -\frac{6}{5} & 1 \end{bmatrix} ;$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{21} & \frac{4}{35} & -\frac{23}{105} & \frac{4}{21} \\ 0 & \frac{1}{10} & \frac{1}{10} & 0 \\ -1 & \frac{7}{5} & -\frac{7}{5} & 0 \\ -\frac{5}{21} & \frac{8}{35} & -\frac{11}{105} & \frac{1}{21} \end{bmatrix};$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{21} & \frac{4}{35} & -\frac{23}{105} & \frac{4}{21} \\ 0 & \frac{1}{10} & \frac{1}{10} & 0 \\ \frac{4}{21} & \frac{9}{35} & -\frac{8}{105} & -\frac{5}{21} \\ -\frac{5}{21} & \frac{8}{35} & -\frac{11}{105} & \frac{1}{21} \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{10} & \frac{1}{10} & 0 \\ \frac{4}{21} & \frac{9}{35} & -\frac{8}{105} & -\frac{5}{21} \\ \frac{1}{21} & \frac{4}{35} & -\frac{23}{105} & \frac{4}{21} \\ -\frac{5}{21} & \frac{8}{35} & -\frac{11}{105} & \frac{1}{21} \end{bmatrix};$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{10} & \frac{1}{10} & 0 \\ \frac{4}{21} & \frac{9}{35} & -\frac{8}{105} & -\frac{5}{21} \\ \frac{1}{21} & \frac{4}{35} & -\frac{23}{105} & \frac{4}{21} \\ -\frac{5}{21} & \frac{8}{35} & -\frac{11}{105} & \frac{1}{21} \end{bmatrix} \begin{bmatrix} 0 \\ 15 \\ 5 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \\ 1 \\ 3 \end{bmatrix}.$$

8. $9x - y = 37,$

$8y - 2z = -4,$

$7z - 3w = -17,$

$2x + \quad = 14;$

$$\begin{bmatrix} 9 & -1 & 0 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 7 & -3 \\ 1 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 37 \\ -2 \\ -17 \\ 7 \end{bmatrix};$$

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$$\begin{bmatrix} 9 & -1 & 0 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 7 & -3 \\ 1 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 9 & -1 & 0 & 0 \\ 0 & 4 & -1 & 0 \\ 1 & 0 & 7 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 9 & -1 & 0 & 0 \\ 36 & 0 & -1 & 0 \\ 1 & 0 & 7 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 9 & -1 & 0 & 0 \\ 36 & 0 & -1 & 0 \\ 253 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 28 & 7 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 9 & -1 & 0 & 0 \\ 36 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ \frac{28}{253} & \frac{7}{253} & \frac{1}{253} & \frac{1}{253} \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 36 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} \frac{1}{253} & -\frac{63}{253} & -\frac{9}{253} & -\frac{9}{253} \\ 4 & 1 & 0 & 0 \\ \frac{28}{253} & \frac{7}{253} & \frac{1}{253} & \frac{1}{253} \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 36 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} \frac{1}{253} & -\frac{63}{253} & -\frac{9}{253} & -\frac{9}{253} \\ 4 & 1 & 0 & 0 \\ \frac{28}{253} & \frac{7}{253} & \frac{1}{253} & \frac{1}{253} \\ -\frac{28}{253} & -\frac{7}{253} & -\frac{1}{253} & \frac{252}{253} \end{bmatrix};$$

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} \frac{1}{253} & -\frac{63}{253} & -\frac{9}{253} & -\frac{9}{253} \\ \frac{4}{253} & \frac{1}{253} & -\frac{36}{253} & -\frac{36}{253} \\ \frac{28}{253} & \frac{7}{253} & \frac{1}{253} & \frac{1}{253} \\ -\frac{28}{253} & -\frac{7}{253} & -\frac{1}{253} & \frac{252}{253} \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \frac{1}{253}\begin{bmatrix} 28 & 7 & 1 & 1 \\ -1 & +63 & +9 & +9 \\ -4 & -1 & 36 & 36 \\ -\frac{28}{3} & -\frac{7}{3} & -\frac{1}{3} & +84 \end{bmatrix};$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \frac{1}{253}\begin{bmatrix} 28 & 7 & 1 & 1 \\ -1 & 63 & 9 & 9 \\ -4 & -1 & 36 & 36 \\ -\frac{28}{3} & -\frac{7}{3} & -\frac{1}{3} & 84 \end{bmatrix}\begin{bmatrix} 37 \\ -2 \\ -17 \\ 7 \end{bmatrix}$$

$$= \frac{1}{253}\begin{bmatrix} 1012 \\ -253 \\ -506 \\ 253 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -2 \\ 1 \end{bmatrix}.$$

3-4. Linear System of Equations

In this chapter we carry out the actual solution of $AX = B$. The techniques involved were developed in the earlier sections and are simply combined here and used to solve systems of linear equations. The parallelism between elementary row operations on the matrices and the procedures used in Section 3-1 should be emphasized.

The discussion about the existence of solutions of systems of linear equations may be amplified to include the ideas of linear dependence. In general a system wherein no equation can be obtained from a linear combination of the remaining equations is said to be a linearly independent system. If it is possible to obtain one of the equations as a linear combination of the remaining equations, then the system is said to be linearly dependent. For instance, the three equations,

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$$
\begin{aligned}
x + 2y - z &= 3, \\
x - y + z &= 4, \\
4x - y + 2z &= 14,
\end{aligned}
$$

which appear on page 120 in this section is a dependent system. Note that if the second equation is multiplied by 3 and added to the first equation, then the result is the third equation. Thus the third equation is a linear combination of the other two.

Generally, when the equations of a system are not linearly independent, there will be more than one solution, and therefore an infinite number of solutions, in the solution set.

Although the language of the chapter must necessarily be complex in order to describe what might be tersely described as "the normal course of events," there are three possible eventualities that the diagonal method will produce:

(a) a contradiction of the form

$$0 = b,$$

where b is not 0, so that the set of equations has no solution;

(b) a unique solution of the form

$$x_i = b_i$$

for all i;

(c) an infinite number of solutions in which some of the variables are expressed as linear combinations of the remaining variables, which might be assigned arbitrary values.

The exercises are routine practice in the solution of simultaneous systems by means of matrix operations.

Exercises 3-4

1. (a) 3 planes parallel.
   2 planes parallel, 3rd intersecting them.
   3 planes collinear.
   2 planes intersecting, 3rd intersecting them but not collinear with them
   3 planes concurrent.

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(b) 3 planes coincident.

2 planes coincident, 3rd parallel to them.

2 planes coincident, 3rd intersecting them.

2(a)
$$\begin{aligned} x + y + 2z &= 1, \\ 2x + \quad z &= 3, \\ 3x + 2y + 4z &= 4; \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -3 \\ 0 & -1 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -3 \\ 0 & -1 & -2 \end{bmatrix}, \begin{bmatrix} -2 & 0 & 1 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -3 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -2 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & -\frac{1}{2} & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -2 & 0 & 1 \\ 10 & 4 & -6 \\ -2 & -\frac{1}{2} & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -2 & 0 & 1 \\ -5 & -2 & 3 \\ +4 & +1 & -2 \end{bmatrix};$$

$$\begin{bmatrix} -2 & 0 & 1 \\ -5 & -2 & 3 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

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2\. (b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix};$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix};$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ +1 & -1 & +1 \\ -1 & 1 & 0 \end{bmatrix};$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}.$$

(c) $x - 2y + z = 1,$
$2x + y - z = 1,$
$x + y + z = 4;$

$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -3 \\ 0 & 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{5} \\ 0 & 5 & -3 \\ 0 & 3 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{5} \\ 0 & 5 & -3 \\ 0 & 0 & \frac{9}{5} \end{bmatrix}, \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & 0 \\ -2 & 1 & 0 \\ \frac{1}{5} & -\frac{3}{5} & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{5} \\ 0 & 1 & -\frac{3}{5} \\ 0 & 0 & \frac{9}{5} \end{bmatrix}, \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & 0 \\ -\frac{2}{5} & \frac{1}{5} & 0 \\ \frac{1}{5} & -\frac{3}{5} & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{9}{5} \end{bmatrix}, \begin{bmatrix} \frac{2}{9} & \frac{1}{3} & \frac{1}{9} \\ -\frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{5} & -\frac{3}{5} & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{2}{9} & \frac{1}{3} & \frac{1}{9} \\ -\frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{9} & -\frac{1}{3} & \frac{5}{9} \end{bmatrix};$$

$$\begin{bmatrix} \frac{2}{9} & \frac{1}{3} & \frac{1}{9} \\ -\frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{9} & -\frac{1}{3} & \frac{5}{9} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix};$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

(d) $$\begin{aligned} 2v + x + y + z &= 0, \\ v - x + 2y + z &= 0, \\ 4v - x + 5y + 3z &= 1, \\ v - x + y - z &= 2; \end{aligned}$$

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & 1 \\ 4 & -1 & 5 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 3 & 0 & 3 & 2 \\ 6 & 0 & 6 & 4 \\ 3 & 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 3 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 3 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix};$$

$$\begin{aligned} 2x + y + z + w &= 0, \\ 3x + 3z + 2w &= 0, \\ 0 &\neq 1, \\ 3x + 2z &= 2. \end{aligned}$$

No solutions.

(e) $$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & -1 \\ 4 & 5 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

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$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 3 & 3 & 2 & 0 \\ 6 & 6 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 3 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 3 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix};$$

$$\begin{aligned} 2x + y + z + w &= 2, \\ 3x + 3y + 2z &= -1, \\ 0 &= 0, \\ 0 &= 0. \end{aligned}$$

An infinite number of solutions. You might, for example, give values to x and y and determine corresponding values for z and w.

3-5. Elementary Row Operations

The purpose of this section is to interpret the row operations in terms of the more fundamental operation of matrix multiplication. We should perhaps stress that matrices of the form J, K, L, are the only ones we call elementary matrices. Their inverses J^{-1}, K^{-1}, L^{-1}, turn out to be matrices of the same form. Some students will be quick to note the form of the product of two elementary matrices, such as

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix},$$

and will be tempted to decompose matrices into factors of the latter form. We feel that such students should be complimented for their insight but must be cautioned that the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

is not an elementary matrix.

Another point that should be stressed is that to perform our row operations we multiply on the left by elementary matrices, and the student must remember that matrix multiplication does not commute. Thus

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

will first multiply the second row by 2 and <u>then</u> add the second row to the third. On the other hand,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

will first add the second row to the third and <u>then</u> multiply the second row by 2. For comparison, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

which are quite different.

The important thing here is for the student to understand that he can perform a row operation on a matrix by left-multiplying the matrix by an elementary matrix, and that all matrices obtained as a result of such multiplications are row equivalent. In particular, any matrix that has an inverse is row equivalent to the inverse.

The exercises are designed to fix the idea of row operation by means of left multiplication of elementary matrices. Exercise 1 is concerned with the determination of some elementary matrices. Exercise 2 asks that a product be decomposed into elementary mat[illegible]ote that in this exercise the answer is not unique, but t[illegible] multiplying will determine the nature of the elementary ma[illegible] Exercise 3 is designed to show the stu[illegible] that if he can find a set of elementary matrix factors of a given matrix he can find the inverse of the given matrix by finding the product of the inverses of the elementary matrices. These matrices can be written by inspection, although the order of the multiplications must be reversed in accordance with

$$A^{-1} = (E_1 \cdot E_2 \cdot E_3, \ldots, E_n)^{-1} = E_n^{-1}, \ldots, E_3^{-1} \cdot E_2^{-1} \cdot E_1^{-1}.$$

Exercise 4 leads the student toward the generalization of the concepts we have already studied in the case of 3×3 matrices. Exercises 8 and 9 are designed to lead the student into a consideration of column operations.

<u>Exercises 3-5</u>

1. (a) $A \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -1 & -1 \end{bmatrix},$

   $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$

   (b) $B \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix},$

   $B = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

(c) $C\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 5 \\ -1 & 0 & 2 \\ 0 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 5 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 0 & 7 \\ 0 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 \\ \frac{1}{7} & \frac{1}{7} & \frac{2}{7} \\ 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 \\ \frac{1}{7} & \frac{1}{7} & \frac{2}{7} \\ -\frac{1}{7} & -\frac{1}{7} & \frac{5}{7} \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{2}{7} & -\frac{5}{7} & \frac{4}{7} \\ \frac{1}{7} & \frac{1}{7} & \frac{2}{7} \\ -\frac{1}{7} & -\frac{1}{7} & \frac{5}{7} \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \frac{2}{7} & -\frac{5}{7} & \frac{4}{7} \\ \frac{1}{7} & \frac{1}{7} & -\frac{5}{7} \\ \frac{1}{7} & \frac{1}{7} & \frac{2}{7} \end{bmatrix} = C.$$

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2\. (a) $$\begin{bmatrix}1&0&0\\0&-1&0\\0&-\frac{3}{8}&-\frac{1}{8}\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&\frac{1}{8}\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$$

The solution here is not unique. There are others. For instance,

$$\begin{bmatrix}1&0&0\\0&\frac{8}{3}&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&\frac{3}{8}&0\\0&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-\frac{1}{8}\end{bmatrix}.$$

The same rema hold 2(b) and 2(c).

(b) $$\begin{bmatrix}1&0&0\\0&1&-3\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-\frac{1}{3}\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&3\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}.$$

(c) $$\begin{bmatrix}1&-1&1\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}.$$

3\. (a) $$\begin{bmatrix}1&0&0\\0&-1&0\\0&-\frac{3}{8}&-\frac{1}{8}\end{bmatrix}^{-1}=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&8\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0\\0&-1&0\\0&3&-8\end{bmatrix}.$$

(b) $$\begin{bmatrix}1&0&0\\0&1&-3\\0&0&1\end{bmatrix}^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&-1\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-3\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0\\0&1&3\\0&0&1\end{bmatrix}.$$

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(c) $$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4. J type: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, multiplies the second row by n.

$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, adds the second row to the first row.

L type: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, interchanges the second and fourth rows.

5. $$\begin{aligned} x - y - 2z &= 3, \\ y + 3z &= 5, \\ 2x + 2y - 3z &= 15; \end{aligned}$$

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 3 \\ 2 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 15 \end{bmatrix};$$

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 3 \\ 2 & 2 & -3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{9}{11} & \frac{7}{11} & \frac{1}{11} \\ -\frac{6}{11} & -\frac{1}{11} & \frac{3}{11} \\ \frac{2}{11} & \frac{4}{11} & -\frac{1}{11} \end{bmatrix};$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{9}{11} & \frac{7}{11} & \frac{1}{11} \\ -\frac{6}{11} & -\frac{1}{11} & \frac{3}{11} \\ \frac{2}{11} & \frac{4}{11} & -\frac{1}{11} \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ 15 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix}.$$

6. (a)
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & -1 \\ -4 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & -1 \\ 4 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 1 \\ 4 & -1 & -2 \\ 2 & 0 & -1 \end{bmatrix}.$$

(b)
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 1 \\ 4 & -1 & -2 \\ 2 & 0 & -1 \end{bmatrix}.$$

(c) The answer to 2(b) is not unique. The order in which certain of the operations may be carried out is arbitrary.

7. By Theorem 3-1, we know that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

This is the first step in the proof by induction.

Suppose we know that

$$(AB\cdots J)^{-1} = J^{-1}\cdots B^{-1}A^{-1}.$$

Then

$$(AB\cdots JK)^{-1} = ((AB\cdots J)K)^{-1}$$

by the associative property for matrix multiplication, whence

$$(AB\cdots JK)^{-1} = K^{-1}(AB\cdots J)^{-1}$$

by Theorem 3-1, so that

$$(AB\cdots JK)^{-1} = K^{-1}(J^{-1}\cdots B^{-1}A^{-1})$$

by the induction hypothesis. Accordingly, we have

$$(AB\cdots JK)^{-1} = K^{-1}J^{-1}\cdots B^{-1}A^{-1},$$

and the induction is complete.

8. (a) $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2a & b & c \\ 2d & e & f \\ 2g & h & i \end{bmatrix}.$

(b) $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & a+c \\ d & e & d+f \\ g & h & g+i \end{bmatrix}.$

(c) $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3a & 2b+c & c \\ 3d & 2e+f & f \\ 3g & 2h+i & i \end{bmatrix}.$

9. Right multiplication of matrices by matrices formed from the products of elementary matrices produces changes in columns similar to the changes made in rows by left multiplication by elementary matrices.

3-6. Summary

This section quite deliberately is somewhat more than a summary. In it, the student reviews the procedures introduced in the chapter, but with slight variations calculated to give him a true mastery of the techniques involved.

The method developed earlier in the chapter is here called the "diagonalization method" and is contrasted with the "triangularization method You might stress these phrases to the class, along with the suggestive word, "pivot."

The triangularization method is exemplified by system III, and the diagonalization method by system IV, on page 104.

You might point out to the class that the triangularization method is an excellent systematic method for solving a single set of linear equations. (The diagonalization method is usually more efficient when two or more systems with a common matrix of coefficients are involved.) Thus, the last equation in system III might be solved for z, the result substituted in the second equation to yield a value for y, and then the two values substituted in the first equation to determine a value for x.

You should also note the streamlining of the diagonalization method through the introduction of complete pivoting. This notion was possibly too involved for the class at the start of the chapter, but it should be quite easy at the end.

For review exercises, you might reassign some of the problems already assigned; but this time have the class use the triangularization method and the streamlined diagonalization method.

Chapter 4

REPRESENTATION OF COLUMN MATRICES AS GEOMETRIC VECTORS

4-1. The Algebra of Vectors

In this chapter, a considerable change is made in the nature of the subject matter. Although it may be helpful to study Chapter 2 and Chapter 3 before Chapter 4, this is not necessary since the present material is largely independent of those two chapters. Students will be able to handle the material of Chapter 4 if they are proficient in the operations of matrix addition and multiplication. In fact, Chapters 1 and 4 together make a worthwhile unit if time is limited.

In Chapter 4, the subject of vectors is introduced. The pace is gentle at the beginning in order to allow sufficient time for the students to become familiar with this new mathematical concept. In Chapter 5, linear transformations are introduced. The material of the latter chapter is considerably more difficult to comprehend, and a teacher should not contemplate handling Chapter 5 unless the class can easily handle the material of Chapter 4. Also, the time needed to understand and complete Chapter 5 is greater than the time necessary to handle Chapter 4 adequately.

In Chapter 4, the exercises vary from simple and straightforward to difficult and abstract. Care should be taken in assigning these problems. Some of them will extend even the most capable students, particularly the exercises dealing with n-dimensional vectors.

In Section 4-1, we consider a special set of column matrices, namely the 2×1 matrices $\begin{bmatrix} a \\ b \end{bmatrix}$, where a and b are real numbers. By definition, such a matrix is called a <u>column vector</u> of order 2. Since these vectors actually are matrices, all the familiar rules pertaining to matrix operations hold for them, as summarized in the two theorems stated in this section. Although the information contained in the theorems is familiar, it is well to review the theorems at length and to do all the exercises involving these theorems in order to establish clearly in the students' minds that column vectors are matrices.

If time is short or the ability of the class is modest, it is better not even to begin a discussion of column vectors of order n. While able students will be challenged by the concept of an n-dimensional vector, and it is relatively easy to work with the algebraic properties of these vectors, a difficulty arises with their geometric interpretation or representation. This stresses the fact that it is important to continually assert that the column vectors or matrices have an algebraic life of their own quite independent of any geometric interpretation.

Exercises 4-1

1. I. (a) $V + W = W + V$:

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

(b) $(V + W) + U = V + (W + U)$:

$$\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) + \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix};$$

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \left(\begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}.$$

(c) $V + 0_{2\times 1} = V$:

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

(d) $V + (-V) = 0_{2\times 1}$:

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0_{2\times 1}.$$

II. (a) $r(V + W) = rV + rW$:

$$2\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) = 2\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \end{bmatrix};$$

$$2\begin{bmatrix} 3 \\ 4 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \end{bmatrix}.$$

(b) r(sV) = (rs)V:

$$2\left((-1)\begin{bmatrix}3\\4\end{bmatrix}\right) = 2 \text{ [illegible] } = \begin{bmatrix}-6\\-8\end{bmatrix},$$

$$(2)(-1)\begin{bmatrix}3\\4\end{bmatrix} = -2\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}-6\\-8\end{bmatrix}.$$

(c) (r + s)V = rV + sV:

$$(2 + (-1))\begin{bmatrix}3\\4\end{bmatrix} = (1)\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix};$$

$$2\begin{bmatrix}3\\4\end{bmatrix} + (-1)\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}6\\8\end{bmatrix} + \begin{bmatrix}-3\\-4\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}.$$

(d) 0V = $\underline{0}$.

$$0\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = \underline{0}.$$

(e) 1V = V:

$$(1)\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}.$$

(f) r0 = 0:

$$2\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = 0.$$

III. (a) A(V + W) = AV + AW:

$$\begin{bmatrix}3 & 0\\2 & -1\end{bmatrix}\left(\begin{bmatrix}3\\4\end{bmatrix} + \begin{bmatrix}-2\\1\end{bmatrix}\right) = \begin{bmatrix}3 & 0\\2 & -1\end{bmatrix}\begin{bmatrix}1\\5\end{bmatrix} = \begin{bmatrix}3\\-3\end{bmatrix};$$

$$\begin{bmatrix}3 & 0\\2 & -1\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix} + \begin{bmatrix}3 & 0\\2 & -1\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}9\\2\end{bmatrix} + \begin{bmatrix}-6\\-5\end{bmatrix} = \begin{bmatrix}3\\-3\end{bmatrix}.$$

(b) $(A + B)V = AV + BV$:

$$\left(\begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\right) \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 16 \\ 0 \end{bmatrix};$$

$$\begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix} + \begin{bmatrix} 7 \\ -2 \end{bmatrix} = \begin{bmatrix} 16 \\ 0 \end{bmatrix}.$$

(c) $A(BV) = (AB)V$:

$$\begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}\left(\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 7 \\ -2 \end{bmatrix} = \begin{bmatrix} 21 \\ 16 \end{bmatrix};$$

$$\left(\begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\right)\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 21 \\ 16 \end{bmatrix}.$$

(d) $\underline{0}V = \underline{0}$:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \underline{0}.$$

(e) $IV = V$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = V.$$

(f) $A(rV) = (rA)V = r(AV)$:

$$\begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}\left(2\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 18 \\ 4 \end{bmatrix};$$

$$\left(2\begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}\right)\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 18 \\ 4 \end{bmatrix};$$

$$2\left(\begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = 2\begin{bmatrix} 9 \\ 2 \end{bmatrix} = \begin{bmatrix} 18 \\ 4 \end{bmatrix}.$$

2. Since addition is associative and commutative, we can rewrite

$$AV - AW = AW + BW$$

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in the (possibly simpler) form

$$AV = 2(AW) + BW.$$

Further, since

$$\delta(A) = -14 \neq 0,$$

A^{-1} exists, and we have

$$A^{-1}AV = 2(A^{-1}A)W + (A^{-1}B)W,$$

or

$$V = 2W + (A^{-1}B)W.$$

Now,

$$A^{-1} = \begin{bmatrix} \frac{1}{7} & \frac{1}{14} \\ \frac{2}{7} & -\frac{5}{14} \end{bmatrix},$$

so that

$$\begin{aligned} V &= \begin{bmatrix} 6 \\ 18 \end{bmatrix} + \left(\begin{bmatrix} \frac{1}{7} & \frac{1}{14} \\ \frac{2}{7} & -\frac{5}{14} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix} \right) \begin{bmatrix} 3 \\ 9 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 18 \end{bmatrix} + \begin{bmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{12}{7} & -\frac{5}{7} \end{bmatrix} \begin{bmatrix} 3 \\ 9 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 18 \end{bmatrix} + \begin{bmatrix} \frac{6}{7} \\ -\frac{9}{7} \end{bmatrix} \\ &= \begin{bmatrix} 6\frac{6}{7} \\ 16\frac{5}{7} \end{bmatrix}, \text{ or } \begin{bmatrix} \frac{48}{7} \\ \frac{117}{7} \end{bmatrix}. \end{aligned}$$

3. We can first solve the given equation for V, thus:

$$2V + 2W = AV + BV,$$
$$2W = AV + BV - 2V,$$
$$= (A + B - 2I)V,$$
$$V = 2(A + B - 2I)^{-1} W.$$

We now compute

$$A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A + B - 2I = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix},$$

$$(A + B - 2I)^{-1} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix},$$

$$2(A + B - 2I)^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = (-1)I,$$

$$V = (-1)IW$$
$$= -W = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

4. $$A(3V = A(BV),$$
$$A^{-1} A(3V) = A^{-1} A(BV),$$
$$I(3V) = I(BV),$$
$$(I3)V = (IB)V,$$
$$(3I - IB)V = \underline{0},$$
$$B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$
$$3I = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix},$$
$$(3I - IB) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \underline{0},$$

$$1v_1 - 1v_2 = 0,$$

$$-1v_1 + 1v_2 = 0,$$

$$v_1 = v_2,$$

$$v = \begin{bmatrix} v_1 \\ v_1 \end{bmatrix}, \quad V_1 \in R.$$

5. (a) $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}.$

(b) $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}.$

(c) If $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $a_{11} = 0$, $a_{21} = 0$.

If $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $a_{12} = 0$, $a_{22} = 0$.

(d) Theorem. If A is a matrix of order 2, and if for every column vector V of order 2 we have

$$AV = O_{2\times 1},$$

then A is the zero matrix $O_{2\times 2}$.

6. (a) Theorem. If A is a matrix of order n, and if for every column vector V of order n we have

$$AV = O_{n\times 1},$$

then A is the zero matrix $O_{n\times n}$.

(b) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

Now

$$A\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}a_{11}\\a_{21}\\a_{31}\end{bmatrix}, \quad A\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}a_{12}\\a_{22}\\a_{32}\end{bmatrix}, \quad A\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}a_{13}\\a_{23}\\a_{33}\end{bmatrix}.$$

If $\begin{bmatrix}a_{11}\\a_{21}\\a_{31}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, then $a_{11} = 0$, $a_{21} = 0$, $a_{31} = 0$;

if $\begin{bmatrix}a_{12}\\a_{22}\\a_{32}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, then $a_{12} = 0$, $a_{22} = 0$, $a_{32} = 0$;

and if $\begin{bmatrix}a_{13}\\a_{23}\\a_{33}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, then $a_{13} = 0$, $a_{23} = 0$, $a_{33} = 0$.

Hence A is the zero matrix.

(c) Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$.

Now

$$A\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} = \begin{bmatrix}a_{11}\\\vdots\\a_{1n}\end{bmatrix}, \quad A\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} = \begin{bmatrix}a_{21}\\\vdots\\a_{2n}\end{bmatrix}, \ldots, \quad A\begin{bmatrix}0\\\vdots\\1\end{bmatrix} = \begin{bmatrix}a_{1n}\\\vdots\\a_{nn}\end{bmatrix}.$$

If $$\begin{bmatrix} a_{11} \\ \vdots \\ a_{1n} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$ then $a_{11} = 0$, $a_{12} = 0, \ldots, a_{1n} = 0$;

$$\vdots$$

and if $$\begin{bmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$ then $a_{1n} = 0, \ldots, a_{nn} = 0$.

Hence A is the zero matrix.

7. We are given that $AV = V$ for every V. This gives us a great deal of freedom to attempt to simplify the problem. Let us look for some V that will make an easy computation (cf. G. Pólya, How To Solve It, Anchor Publishing Co., paperback).

(i) $V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ makes everything easy, but unfortunately gives us no information.

(ii) $V = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ gives us this:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

At this stage we know that A must be

$$\begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix},$$

but we do not yet know what restrictions, if any, must be put on b and d.

(iii) Let's try another V. If $AV = V$ holds for all V, then, in particular, it must hold for $V = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

This gives us

$$\begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

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$$\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and we now know that A must be

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Theorem. If A is a matrix of order 2, and if for every column vector V of order 2 we have

$$AV = V,$$

then $A = I$.

Note: This leaves one question unanswered. We have found that $A = I$ is a necessary condition for $AV = V$. Is it also sufficient? Yes, it is. We prove this as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

as we discover by multiplying out IV.

We could, consequently, strengthen our theorem to state that $AV = V$ for all V if and only if $A = I$.

8. Theorem. If A is a square matrix of order n, and if for every column vector V of order n we have

$$AV = V,$$

then $A = I$.

Proof. Let

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

Now

$$A\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} \quad\text{and}\quad A\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} = \begin{bmatrix}a_{11}\\a_{21}\\\vdots\\a_{n1}\end{bmatrix};$$

hence

$$a_{11} = 1, \quad a_{21} = a_{31} = \cdots = a_{n1} = 0.$$

Also

$$A\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} \quad\text{and}\quad A\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} = \begin{bmatrix}a_{12}\\a_{22}\\\vdots\\a_{n2}\end{bmatrix};$$

hence

$$a_{12} = 0, \quad a_{22} = 1, \quad a_{32} = a_{42} = \cdots = a_{n2} = 0.$$

Continuing, we see that

$$a_{ij} = 1 \quad \text{for all} \quad i = j,$$
$$a_{ij} = 0 \quad \text{for all} \quad i \neq j.$$

9. Theorem 4-1'. Let V and W be row vectors of order 2, and let A be a square matrix of order 2. Let r be a number. Then

$$V + W, \quad rV, \quad \text{and} \quad VA$$

are each column vectors of order 2.

Theorem 4-2'. This is identical with Theorem 4-2 except that the word "column" must be changed to "row", and products of the form

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix},$$

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whenever they occur, must be changed to

$$x_1\ x_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This means the following changes:

II: (a) becomes $(V + W)A = VA + WA$,

(b) becomes $V(A + B) = VA + VB$,

(c) becomes $(VA)B = V(AB)$,

(d) becomes $V\,0_2 = 0_{1 \times 2}$,

(e) becomes $VI = V$,

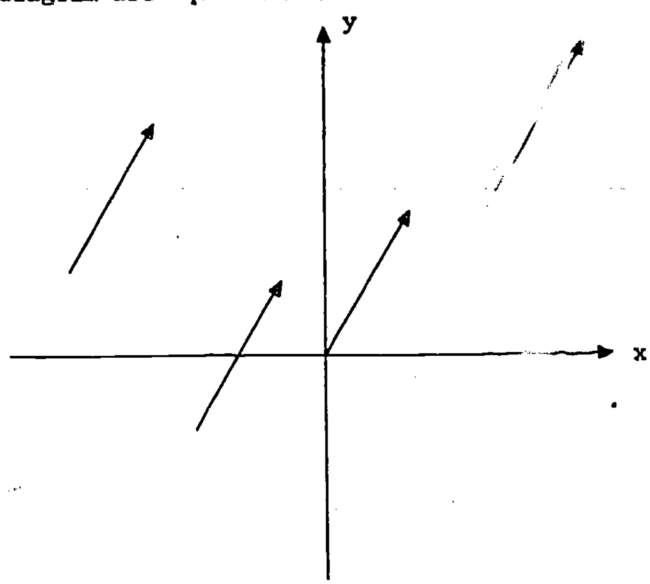
(f) becomes $(rV)A = V(rA) = r(VA)$.

Finally, to show the isomorphism, map every matrix into its transpose, write every column vector as a row vector, and reverse the order of products as we have just done above.

4-2. Vectors and their Geometric Representation

In this section, a correspondence is established between the set of all column vectors of order 2 and the set of all located vectors in the plane. As this concept is developed, students are apt to use the term 'vector' loosely and eventually persuade themselves that the column vectors (matrices) and the arrows are one and the same thing. This idea is to be avoided since it limits the power of the vector concept considerably. For instance there is a very important algebraic life for a vector with six entries although this has no geometric representation of an ordinary nature. The geometric representation enables the student to visualize the algebra and helps him comprehend the relationships; it does not 'prove' the theorems of the algebra nor does it limit the operations.

At the beginning of the section, a column vector is associated with a located vector, that is, a vector with a fixed length and a fixed direction and an arbitrary initial point. The two properties of a geometric vector, length and direction, are equally well represented by each element of a set. (The latter is sometimes labeled an equivalence class of vectors.) We say the vectors in the following diagram are equivalent.



The concept of equivalent vectors has great power and is immediately applicable to proofs of familiar plane geometry theorems (See "Geometrical Vectors and the Concept of Vector Space" 23rd Yearbook — National Council of Teachers of Mathematics.)

Since it provides a uniform pattern that will bring about a greater class cohesiveness and yet not sacrifice any desirable power, we quickly introduce a standard representation. By this time, the students will be somewhat familiar with a located vector and the idea can be reintroduced when needed to increase understanding.

Two properties associated with the directed line segments that correspond to column vectors are length and direction. Note that for ~~the column~~ vector

$$V = \begin{bmatrix} u \\ v \end{bmatrix},$$

the symbol $||V||$ stands for the length of the directed line segment; it is equal to the nonnegative square root of $u^2 + v^2$:

$$||V|| = \sqrt{u^2 + v^2} .$$

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The direction of the segment is given by the two direction cosines. In elementary algebra we learn to associate a slope with a line segment; but direction is not specified by giving the slope alone, since a line with a particular slope may have either of two different directions, or not be directed at all. For instance, a line with slope 3/4 may be considered as pointing toward the upper right as well as toward the lower left.

In order to avoid certain inconveniences that would otherwise arise, we arbitrarily choose to say that the zero vector is a directed line segment and that it has the same direction as any other vector. If this were not done, the presentation would not be so elegant, since exceptions would occur.

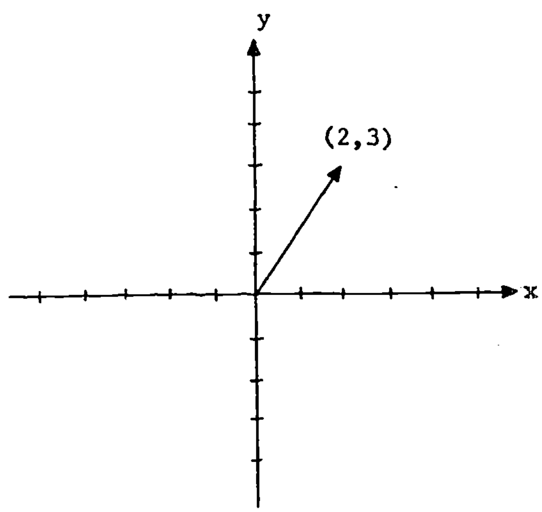
Exercises 4-2

1. Same length: (c), (e), (f).

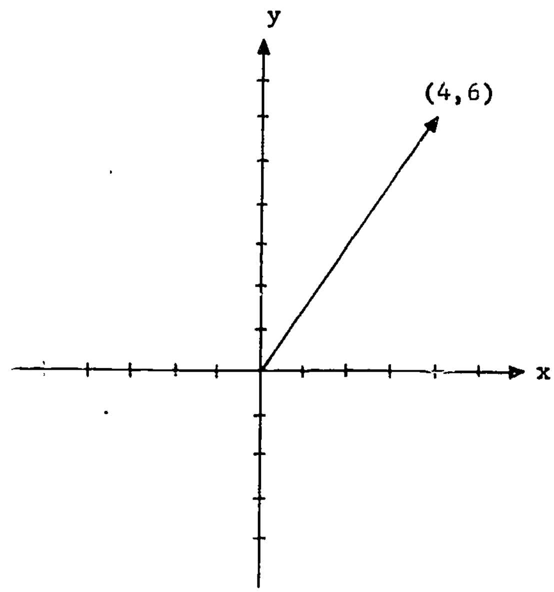
   Same direction: (a), (d) (because the zero vector "has the same direction as any and every other vector"; see p. 165, top), (g), and (j), provided that $t \geq 0$.

   Some pairs lie along the same line, but not along the same ray (i.e., they have "opposite" directions). These are (b), (d) (again, because of the universal direction of the zero vector), and (j) if $t \leq 0$ (note that $t = 0$ also gives us the zero vector).

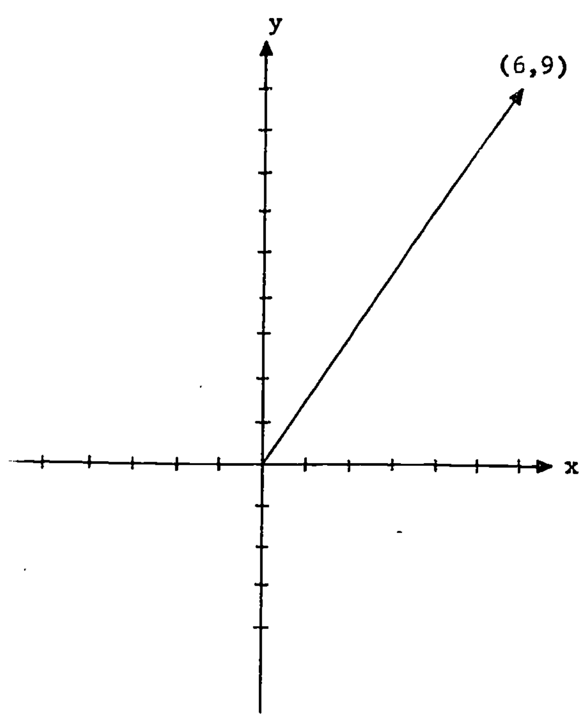
2.



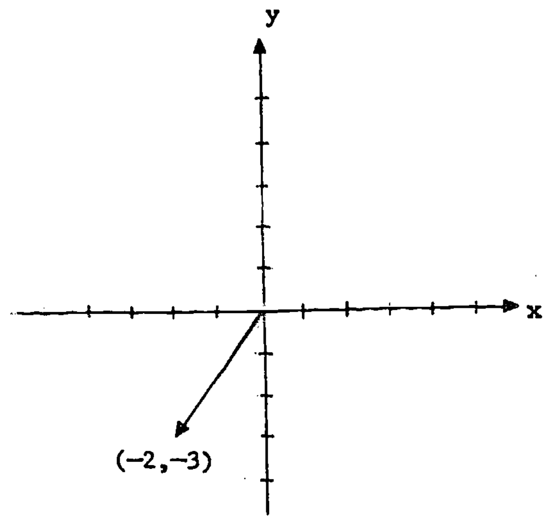
$t = 1$



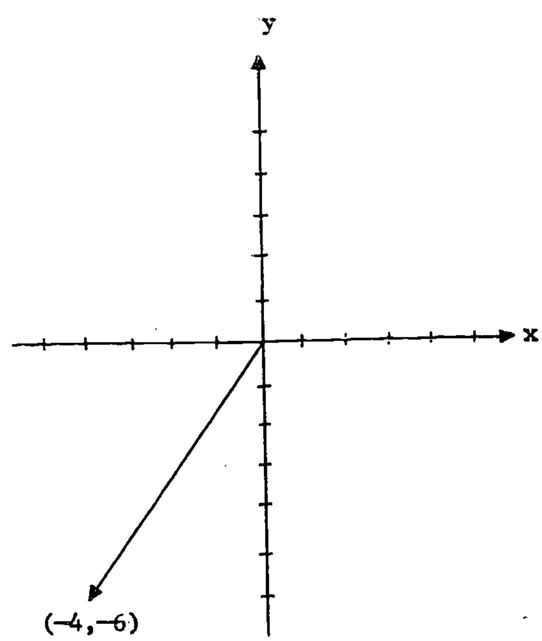
t = 2



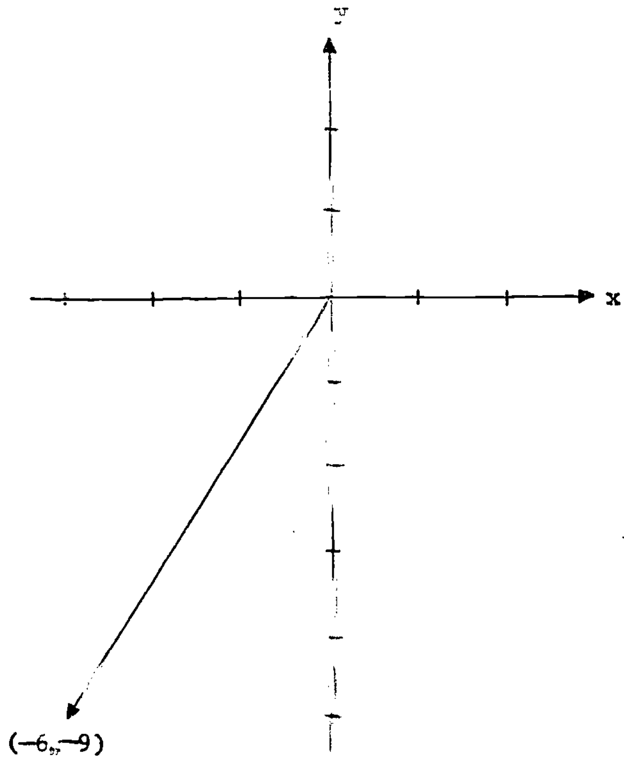
t = 3



$t = -1$

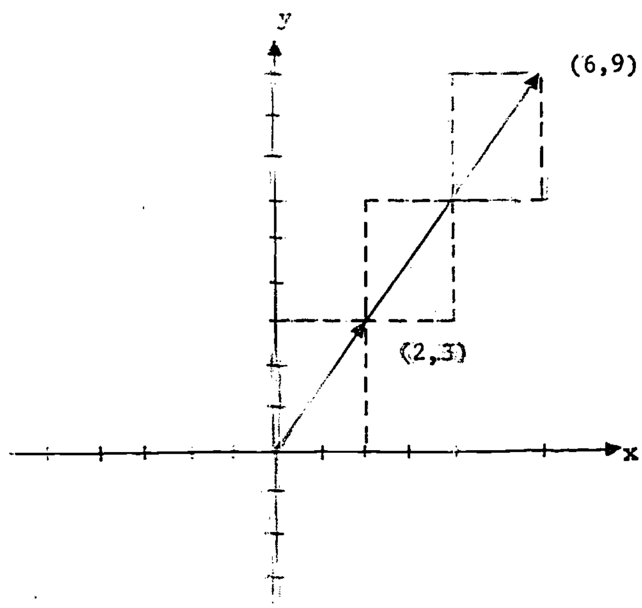


$t = -2$



t = - 3

It is also interesting to compare $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $3\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ in the same diagram:



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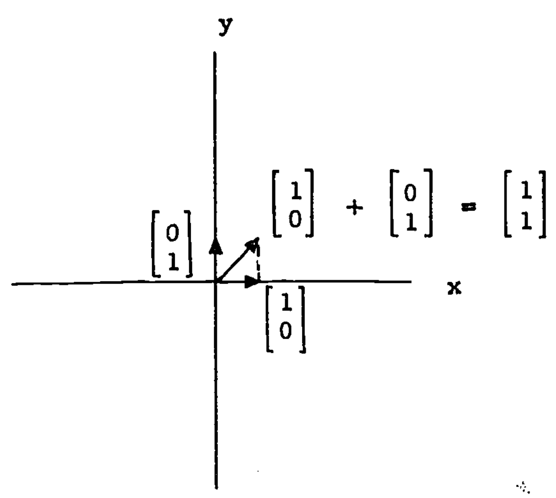
The lengths and direction cosines α, β are as follows:

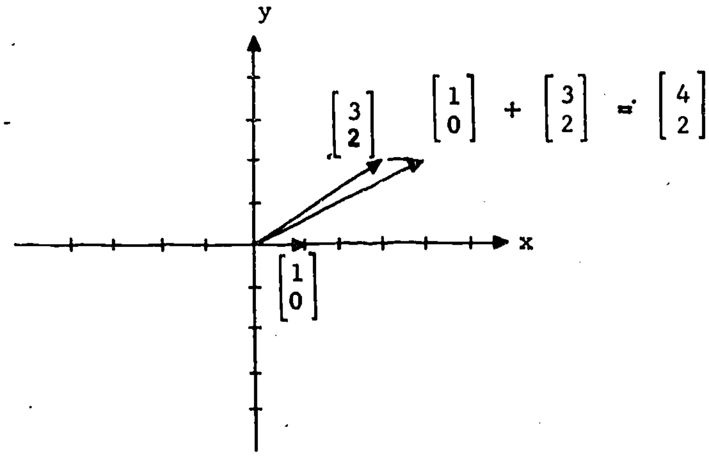
t	Length	α	β
1	$\sqrt{13}$	$2/\sqrt{13}$	$3/\sqrt{13}$
2	$2\sqrt{13}$	$2/\sqrt{13}$	$3/\sqrt{13}$
3	$3\sqrt{13}$	$2/\sqrt{13}$	$3/\sqrt{13}$
−1	$\sqrt{13}$	$-2/\sqrt{13}$	$-3/\sqrt{13}$
−2	$2\sqrt{13}$	$-2/\sqrt{13}$	$-3/\sqrt{13}$
−3	$3/\sqrt{13}$	$-2/\sqrt{13}$	$-3/\sqrt{13}$

3. (a)



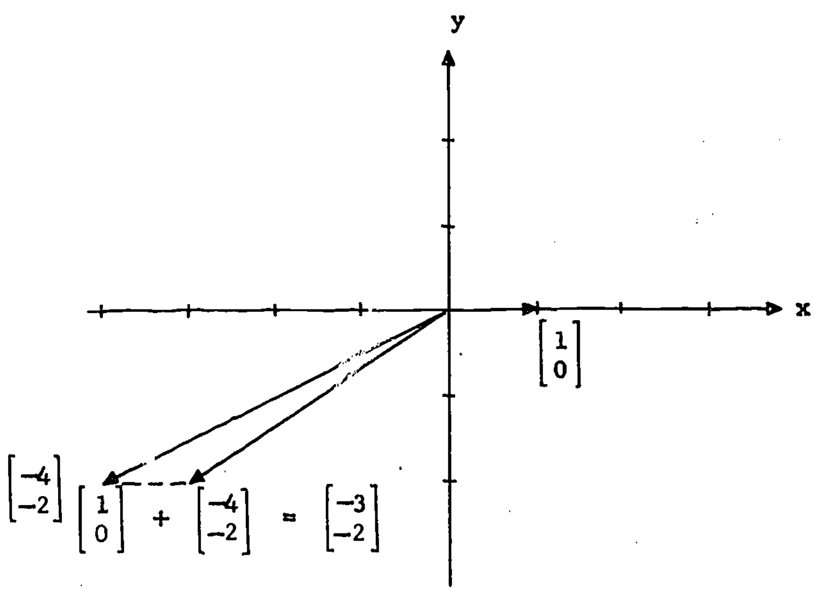
Vector	Length	α	β
$\begin{bmatrix}1\\0\end{bmatrix}$	1	1	0
$\begin{bmatrix}0\\1\end{bmatrix}$	1	0	1
$\begin{bmatrix}1\\1\end{bmatrix}$	$\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$

(b)



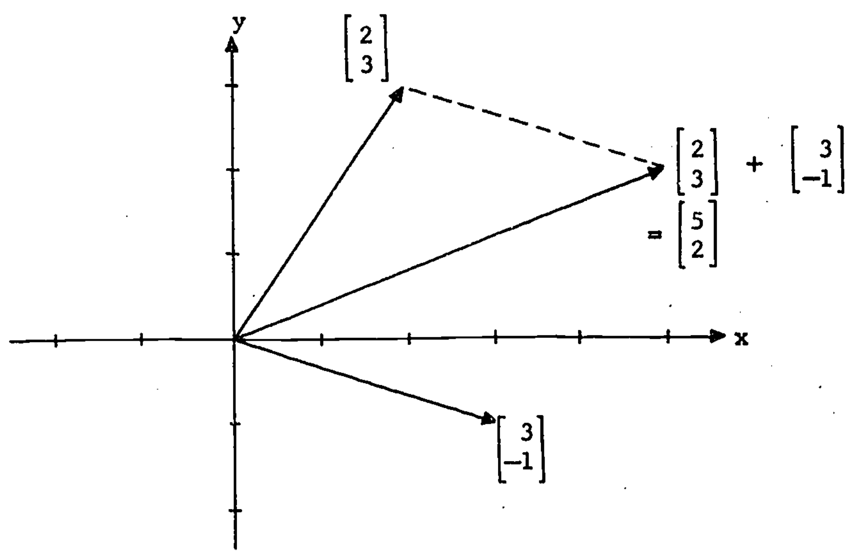
Vector	Length	α	β
$\begin{bmatrix}1\\0\end{bmatrix}$	1	1	0
$\begin{bmatrix}3\\2\end{bmatrix}$	$\sqrt{13}$	$3/\sqrt{13}$	$2/\sqrt{13}$
$\begin{bmatrix}4\\2\end{bmatrix}$	$2\sqrt{5}$	$2/\sqrt{5}$	$1/\sqrt{5}$

(c)



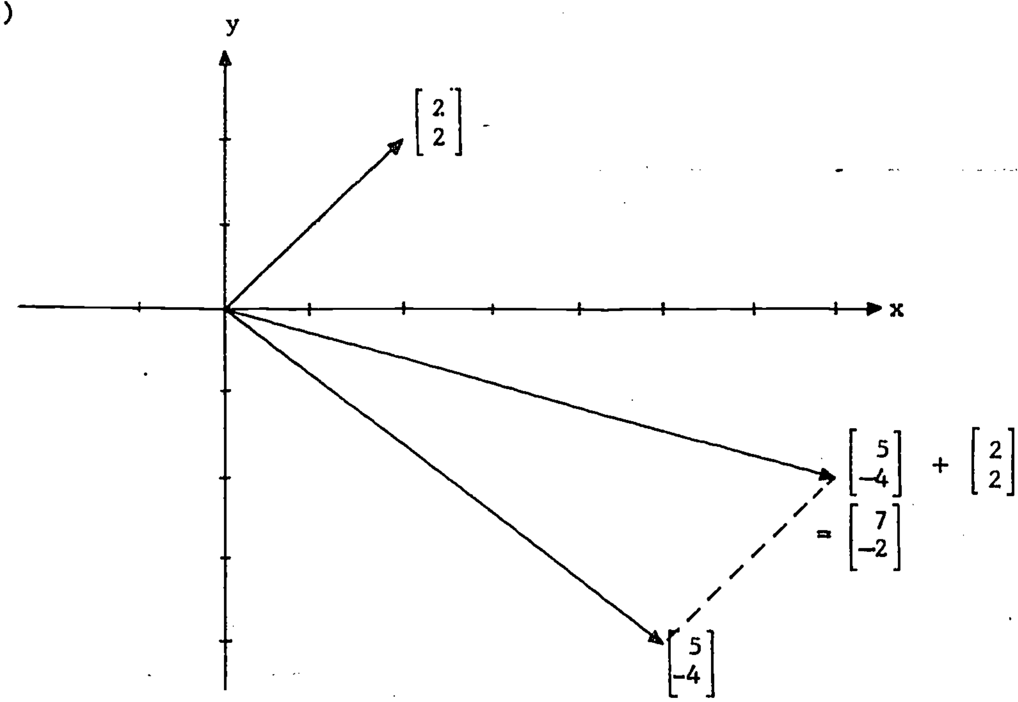
Vector	Length	α	β
$\begin{bmatrix}1\\0\end{bmatrix}$	1	1	0
$\begin{bmatrix}-4\\2\end{bmatrix}$	$2\sqrt{5}$	$-2/\sqrt{5}$	$1/\sqrt{5}$
$\begin{bmatrix}-3\\2\end{bmatrix}$	$\sqrt{13}$	$-3/\sqrt{13}$	$2/\sqrt{13}$

(d)



Vector	Length	α	β
$\begin{bmatrix}2\\3\end{bmatrix}$	$\sqrt{13}$	$2/\sqrt{13}$	$3/\sqrt{13}$
$\begin{bmatrix}3\\-1\end{bmatrix}$	$\sqrt{10}$	$3/\sqrt{10}$	$-1/\sqrt{10}$
$\begin{bmatrix}5\\2\end{bmatrix}$	$\sqrt{29}$	$5/\sqrt{29}$	$2/\sqrt{29}$

(e)



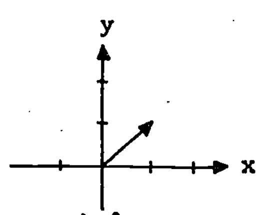
Vector	Length	α	β
$\begin{bmatrix} 5 \\ -4 \end{bmatrix}$	$\sqrt{41}$	$5/\sqrt{41}$	$-4/\sqrt{41}$
$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	$2\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$
$\begin{bmatrix} 7 \\ -2 \end{bmatrix}$	$\sqrt{53}$	$7/\sqrt{53}$	$-2/\sqrt{53}$

4. $V = \begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 1 \\ m \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix}$.

(a) $m = 1, \quad b = 0;$

$$V = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$t = 1; \qquad V = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$



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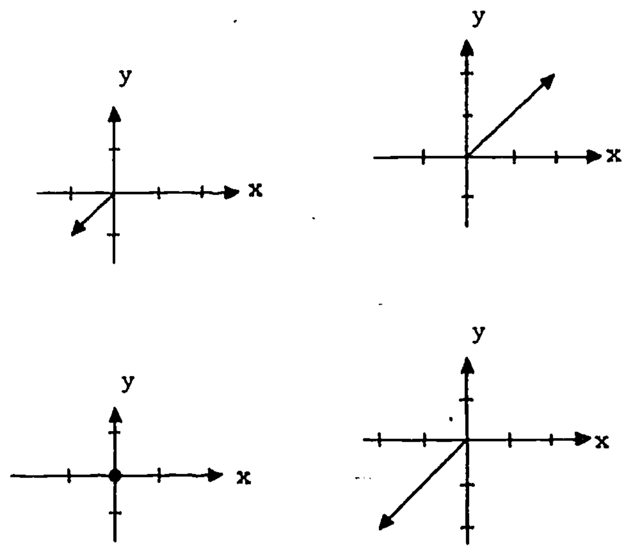
$t = 2; \qquad V = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$

$t = -1; \qquad V = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$

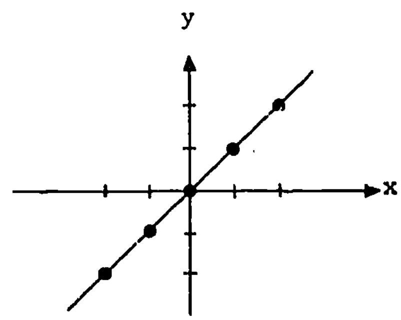
$t = -2; \qquad V = \begin{bmatrix} -2 \\ -2 \end{bmatrix}.$

$t = 0; \qquad V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

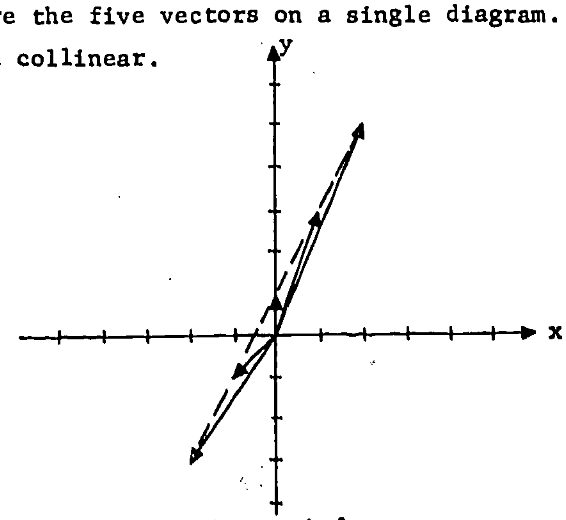


In each case, the point (x,y) is located at the tip of the arrow.

An important theorem tells us that these five points (plus the infinitely many points obtained from other values of t) must all be collinear:



(b) We show here the five vectors on a single diagram. Again, the five "arrow-tips" are collinear.



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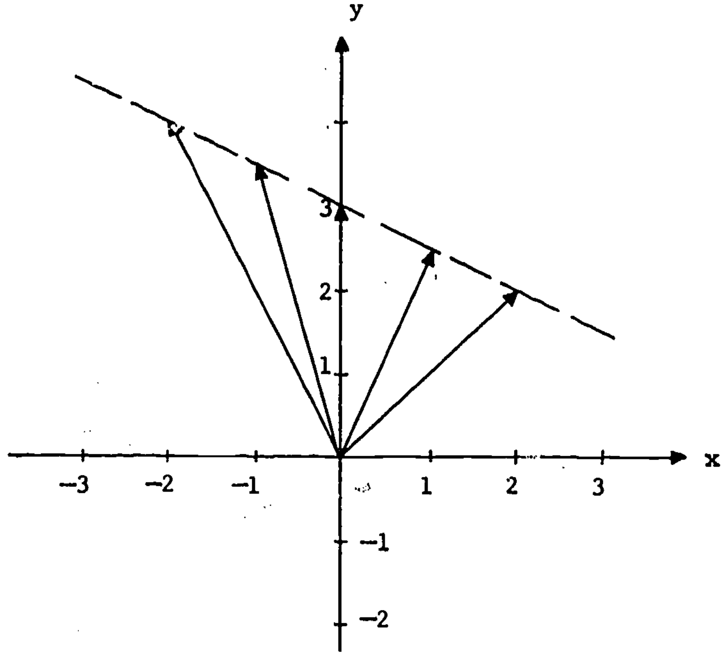
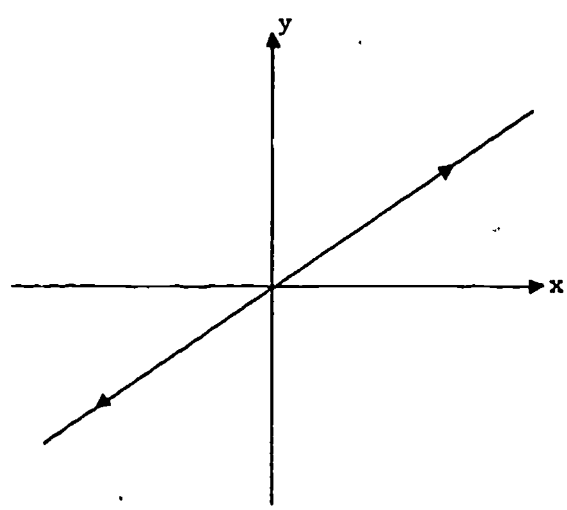
(c)



5. If A and B lie on the same ray from the origin, their direction cosines, evidently, are equal. If A and B do <u>not</u> lie on the same ray from the origin (as in the case shown in the figure),



then they have "opposite" directions; thus if

$$A = \begin{bmatrix} a \\ b \end{bmatrix},$$

then

$$B = \begin{bmatrix} ta \\ tb \end{bmatrix},$$

where $t < 0$. The direction cosines of A, then, are

$$\frac{a}{\sqrt{a^2 + b^2}}, \quad \frac{b}{\sqrt{a^2 + b^2}},$$

and those of B are

$$\frac{t}{|t|}\frac{a}{\sqrt{a^2 + b^2}}, \quad \frac{t}{|t|}\frac{b}{\sqrt{a^2 + b^2}}.$$

Since $t < 0$, $\frac{t}{|t|} = -1$, and the direction cosines of B are the negatives (or "opposites") of those of A.

6. (a) $\begin{bmatrix} u \\ v \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} t \\ 2t \end{bmatrix}$, where $t \in R$.

(b) $t\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -t \\ 3t \end{bmatrix}$.

(c) $t\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} t \\ 3t \end{bmatrix}$, $\left(\text{which, of course, is the same as } \begin{bmatrix} 5t \\ 15t \end{bmatrix}\right)$.

(d) $t\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} t \\ -t \end{bmatrix}$.

(e) $t\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3t \end{bmatrix}$, $\left(\text{which could also be written } \begin{bmatrix} 0 \\ t \end{bmatrix}\right)$.

4-3. Geometrical Interpretation of the Multiplication of a Vector by a Number

The ideas presented in this section can easily be grasped intuitively. To present proofs is somewhat more challenging. This is a situation that occurs frequently in mathematics. Although it may be cumbersome and seem unnecessary to the less sophisticated students, unless we are to extend our postulates or give an intuitive exposition, we must give proofs of our conclusions or theorems. It is always easy to shatter an intuitive conclusion by asking the very proper mathematical question, "Why?"

As stated in Chapter 1, the multiplication of a vector or matrix by a number is frequently referred to as scalar multiplication. This should not be confused with the multiplication of a matrix by a matrix, or with the multiplication of a vector by a vector.

Exercises 4-3

1. (a) $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$.

(b) $\begin{bmatrix} 6 \\ 9 \end{bmatrix}$.

(c) $\begin{bmatrix} -\frac{2}{3} \\ -1 \end{bmatrix}$.

(d) $\begin{bmatrix} 6 \\ x \end{bmatrix}$, where $x \in R$, $x \neq 9$.

(e) $\begin{bmatrix} 8t \\ 12t \end{bmatrix}$.

(f) $\begin{bmatrix} -8t \\ -12t \end{bmatrix}$.

(g) $\begin{bmatrix} h \\ h \end{bmatrix}$ is one possible answer. There are infinitely many others.

Some other correct answers:

$$\begin{bmatrix} h \\ 0 \end{bmatrix}, \quad \begin{bmatrix} h \\ 2h \end{bmatrix}, \quad \begin{bmatrix} h \\ -h \end{bmatrix}, \quad \begin{bmatrix} h \\ 7h \end{bmatrix}, \quad \begin{bmatrix} h \\ 3h^2 + 3 \end{bmatrix}.$$

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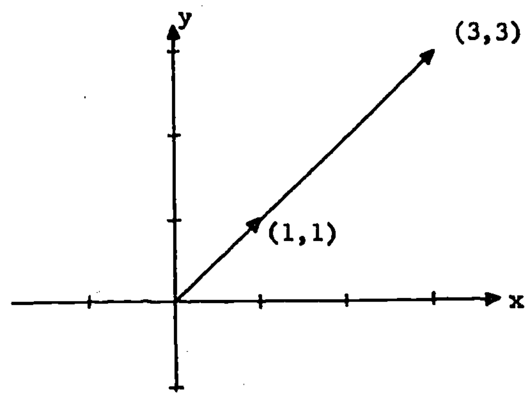
(This last answer is tricky; it is true that $\begin{bmatrix} h \\ 3h^2 + 3 \end{bmatrix} \notin L$, since

$$\frac{h}{3h^2 + 3} = \frac{2}{3}$$

leads to a quadratic equation with no real roots.)

2. (a)

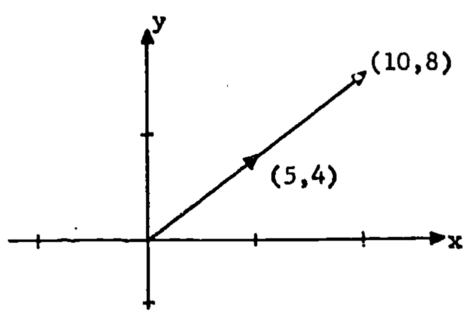


$\alpha = \frac{1}{\sqrt{2}}$,

$\beta = \frac{1}{\sqrt{2}}$;

$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b)

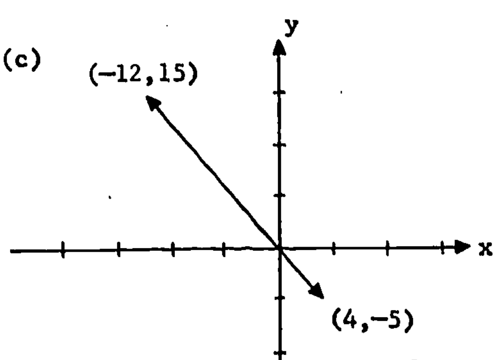


$\alpha = \frac{5}{\sqrt{41}}$,

$\beta = \frac{4}{\sqrt{41}}$;

$\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 10 \\ 8 \end{bmatrix}$.

(c)



$\alpha = \pm\frac{4}{\sqrt{41}}$,

$\beta = \pm\frac{5}{\sqrt{41}}$;

$\begin{bmatrix} -12 \\ 15 \end{bmatrix} = -3\begin{bmatrix} 4 \\ -5 \end{bmatrix}$.

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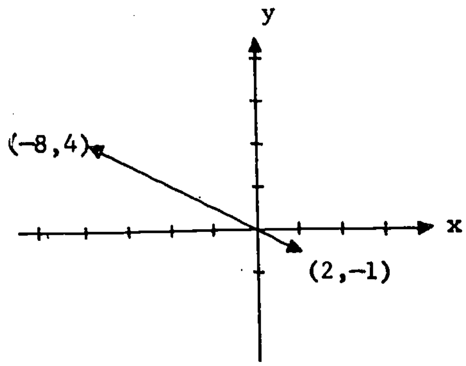
(d) $\begin{bmatrix} 2 \\ -32 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -16 \end{bmatrix}$ are collinear, with $\begin{bmatrix} 2 \\ 32 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 16 \end{bmatrix}$.

(e)

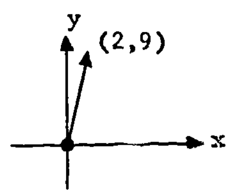


$$\alpha = \pm \frac{2}{\sqrt{5}},$$

$$\beta = \pm \frac{1}{\sqrt{5}};$$

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -8 \\ 4 \end{bmatrix}.$$

(f) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, the zero vector, may be



assigned the direction of any vector;

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 2 \\ 9 \end{bmatrix}.$$

3. Let

$$\frac{v_1}{||V||} \text{ and } \frac{v_2}{||V||}$$

be direction cosines for

$$V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

and let

$$\frac{w_1}{||W||} \text{ and } \frac{w_2}{||W||}$$

be direction cosines for

$$W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} .$$

Since V and W are parallel,

$$\frac{v_1}{||V||} = \frac{w_1}{||W||}, \text{ and } \frac{v_2}{||V||} = \frac{w_2}{||W||} .$$

Hence $\frac{v_1}{w_1} = \frac{v_2}{w_2}$.

Since v_1, v_2, w_1, w_2 are real numbers,

$$\frac{v_1}{w_1} = \frac{v_2}{w_2} = r, \qquad r \in R.$$

Now $v_1 = rw_1$ and $v_2 = rw_2$.

Hence $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} rw_1 \\ rw_2 \end{bmatrix} = r \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, or $V = rW$.

4. (a) Let $V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Hence $rV = r \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} rv_1 \\ rv_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Since $rv_1 = 0$, and $r \neq 0$, we have $v_1 = 0$; similarly, $v_2 = 0$ (from the field axioms for real numbers). Hence,

$$V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(b) Since $rv_1 = 0$, and $v_1 \neq 0$, we have $r = 0$, as in part (a).

5. Let $V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, so that $V + rV = \begin{bmatrix} v_1 + rv_1 \\ v_2 + rv_2 \end{bmatrix} = \begin{bmatrix} (1 + r)v_1 \\ (1 + r)v_2 \end{bmatrix}$.

Direction cosines of V:

$$\alpha = \frac{v_1}{\sqrt{v_1^2 + v_2^2}}, \quad \beta = \frac{v_2}{\sqrt{v_1^2 + v_2^2}}.$$

Direction cosines of $V + rV$:

$$\alpha = \frac{v_1 + rv_1}{\sqrt{(v_1 + rv_1)^2 + (v_2 + rv_2)^2}} = \frac{(1 + r)v_1}{\sqrt{(1 + r)^2(v_1^2 + v_2^2)}} = \pm\frac{v_1}{\sqrt{v_1^2 + v_2^2}},$$

$$\beta = \frac{\pm v_2}{\sqrt{v_1^2 + v_2^2}}.$$

When $r \geq -1$, $1 + r \geq 0$, and the signs are both $+$.

When $r < -1$, $1 + r < 0$, and the signs are both $-$.

$$||V + rV|| = \sqrt{(1 + r)^2(v_1^2 + v_2^2)} = |1 + r|\sqrt{v_1^2 + v_2^2}$$

$$= ||V|| \ |1 + r|.$$

4-4. Geometrical Interpretation of the Addition of Two Vectors

Through their study of physics, many students will be familiar with the parallelogram of forces. In the physical sciences, a force can be represented as an arrow. The length of the arrow indicates the magnitude or strength of the force, and the direction of the arrow indicates the direction of the force. If two forces act on a body at the same point, the effect is the same as though the body were acted on by a single force. The single force, or resultant of the two forces, is represented by the diagonal arrow of the parallelogram having the two arrows representing the original two forces as sides.

When addition was studied in Chapter 1, we learned that the sum of two matrices that are conformable for addition is found by adding together the entries

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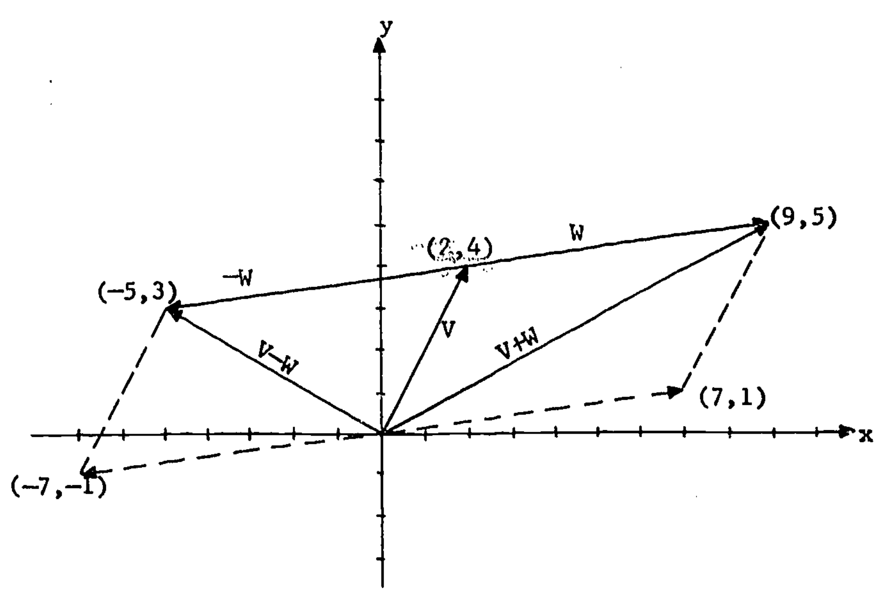
in the same position, pair by pair. When this rule is applied to column vectors of order two, it means that we add the two first entries and then the two second entries. When column vectors are represented as directed line segments from the origin, then the first entry represents the abscissa and the second entry represents the ordinate. The sum of the two vectors is represented by a directed line segment OP such that the coordinates of the point P are the two sums of the respective entries of the original vectors. It is to be noted that no exception need be made if the two vectors are parallel. The key to the proof of the theorem that a parallelogram is formed is the proposition from plane geometry stating that "if the opposite pairs of sides of a quadrilateral are equal, the quadrilateral is a parallelogram."

Exercises 4-4

1. (a) $\begin{bmatrix} 9 \\ 5 \end{bmatrix}$, $\begin{bmatrix} -5 \\ 3 \end{bmatrix}$.



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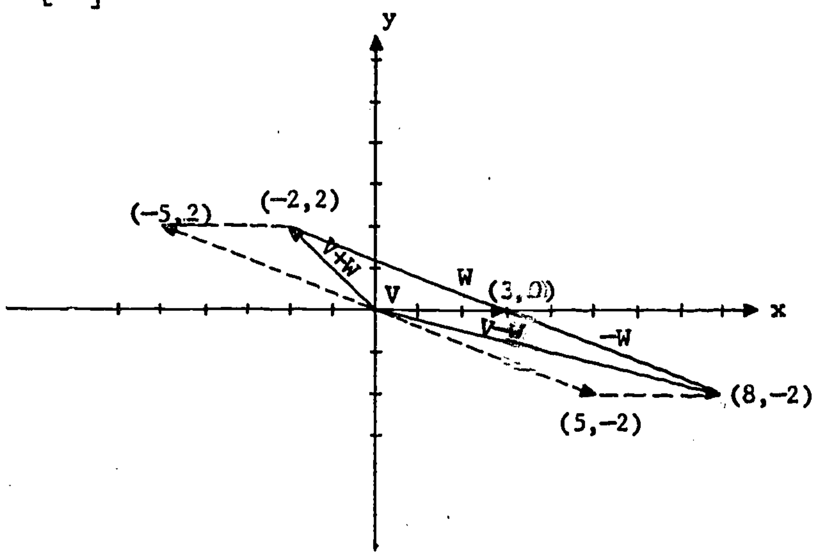
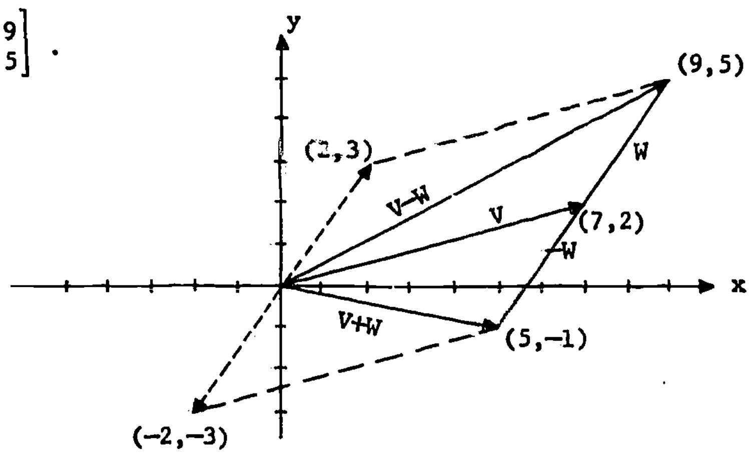
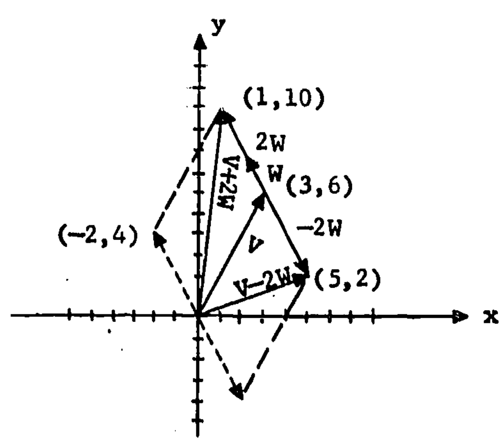
(b) $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 8 \\ -2 \end{bmatrix}$.



(c) $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 9 \\ 5 \end{bmatrix}$.



(d) $\begin{bmatrix} 1 \\ 10 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$.



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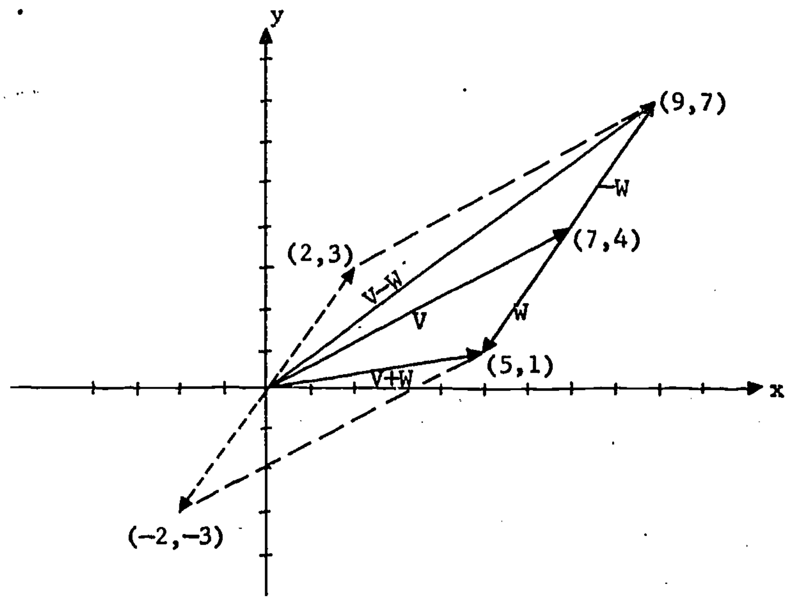
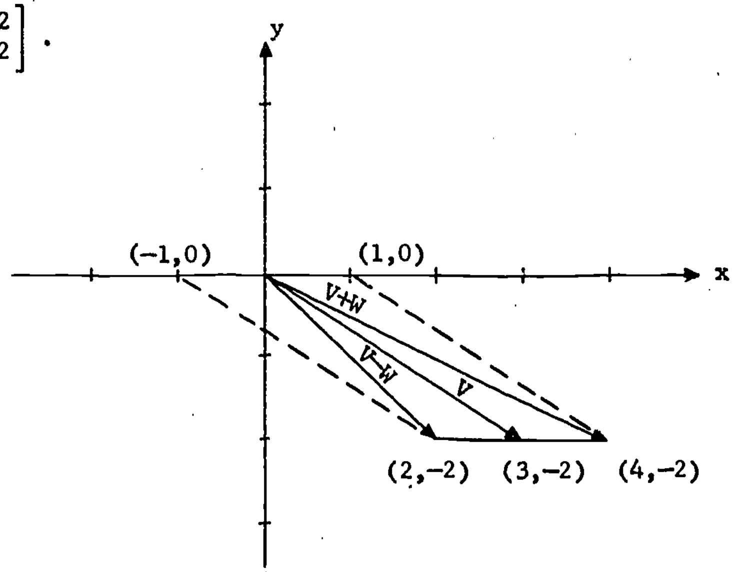
(e) $\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 7 \end{bmatrix}$.

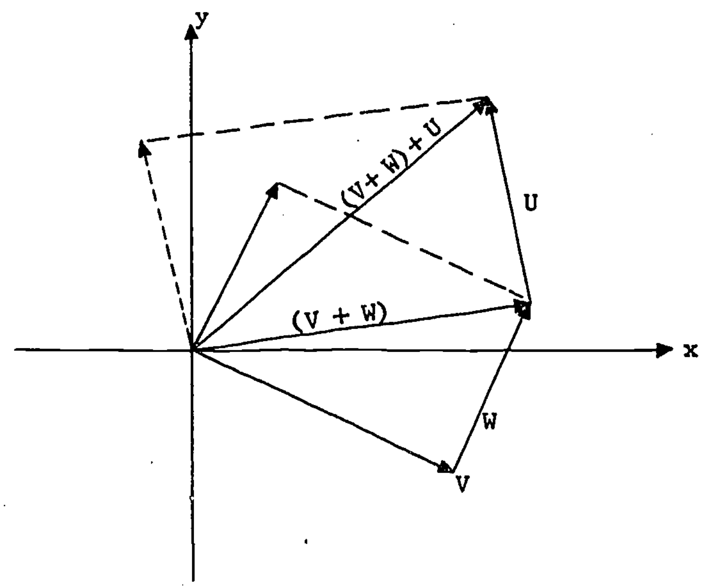


(f) $\begin{bmatrix} 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}$.

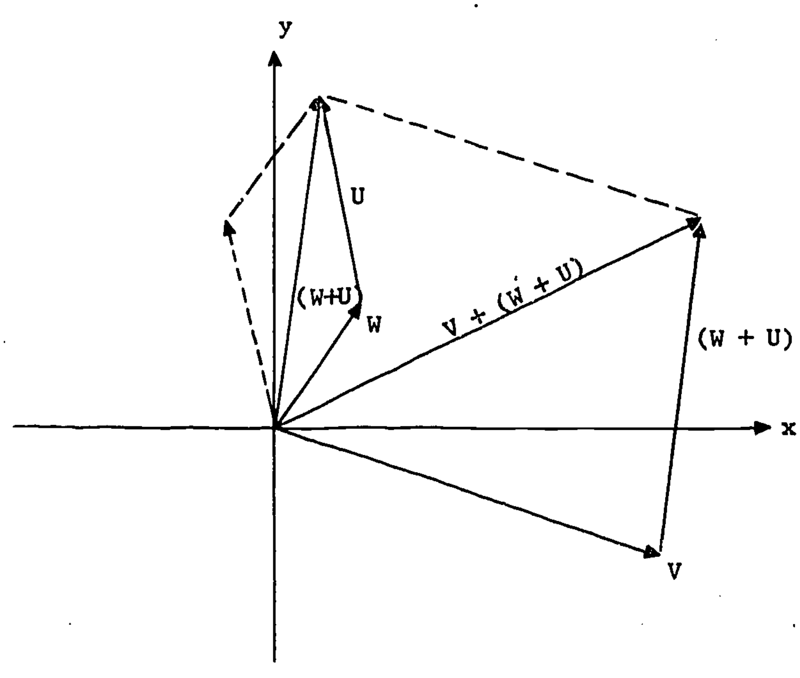


In constructing the sum, the order does not matter. But in constructing the difference the order must not be interchanged.

2. (V + W) + U:



V + (W + U):

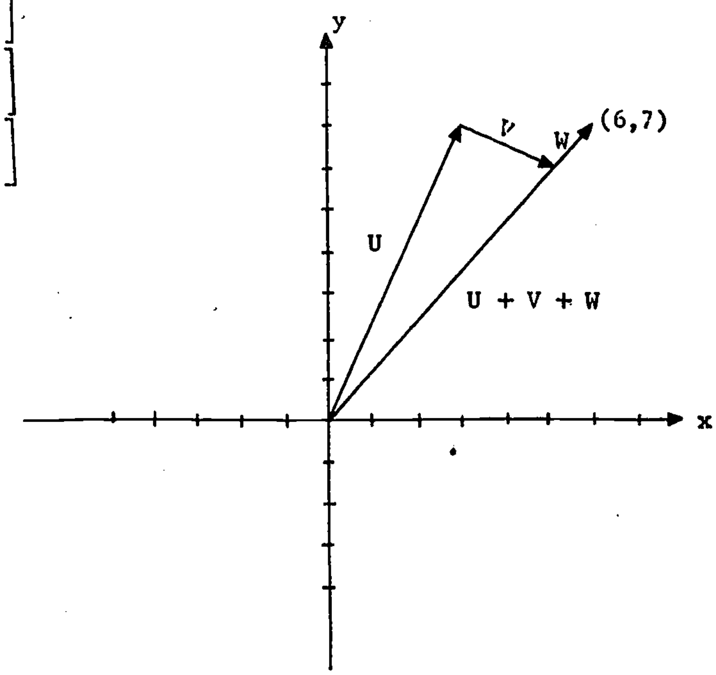


3. (a) $\begin{bmatrix} 6 \\ 7 \end{bmatrix}$.

$U = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$V = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$W = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

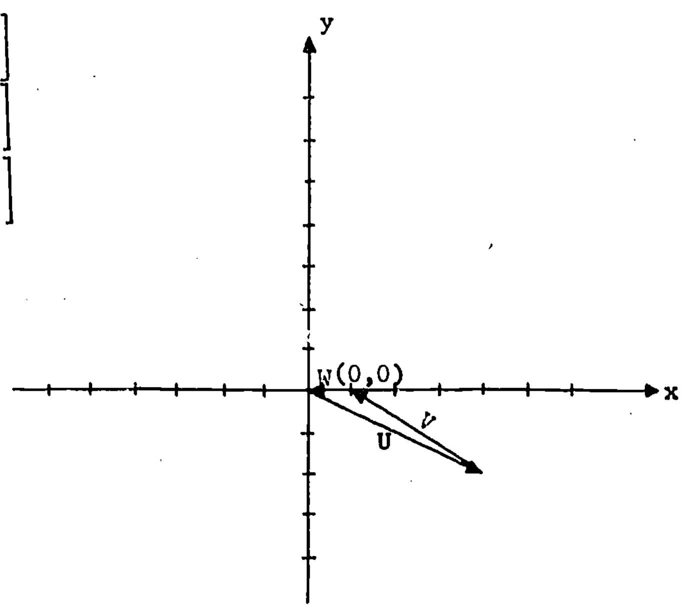


(b) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$U = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

$V = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$W = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$



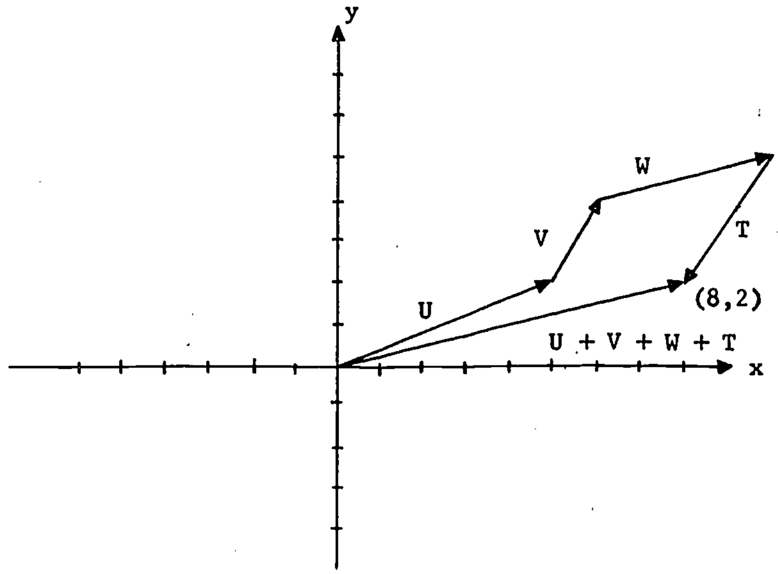
(c) $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$.

$U = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$V = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$W = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$T = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$



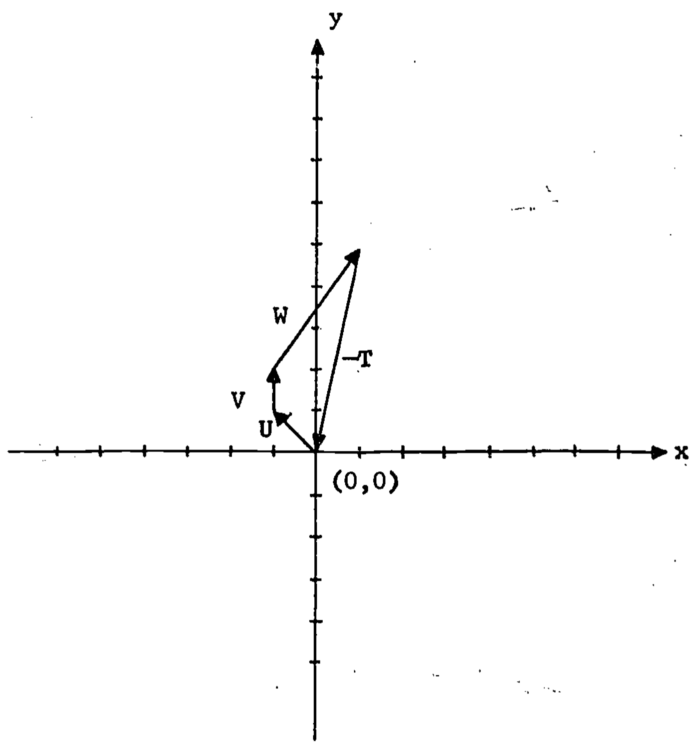
(d) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$U = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$V = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$W = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$T = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$



4. (a) V and W represent two geometric vectors equal in magnitude and opposite in direction.

   (b) A triangle may be constructed from three line segments having the length and direction of the geometric vectors V, W and U.

   (c) A quadrilateral may be constructed from four line segments having the length and direction of the geometric vectors V, W, U, and T.

5. $V = \begin{bmatrix} u \\ v \end{bmatrix}$, $W = \begin{bmatrix} v \\ s \end{bmatrix}$, $V + W = \begin{bmatrix} u + v \\ v + s \end{bmatrix}$;

   $\overrightarrow{OT}$ is the line segment from $(0,0)$ to $(u + r, v + s)$. Length of

   $$\overrightarrow{PR} = \sqrt{(u + r - u)^2 + (v + s - v)^2} = \sqrt{r^2 + s^2} = \text{length of } \overrightarrow{OT}.$$

   $$\text{Slope of } \overrightarrow{PR} = \frac{v + s - v}{u + r - u} = \frac{s}{r} = \text{slope of } \overrightarrow{OT}.$$

4-5. The Inner Product of Two Vectors

When addition was discussed in Chapter 1, the sum of two matrices of the same order was defined. The sum was obtained by adding corresponding entries in the two matrices. At that time, it may have seemed natural to obtain a product by multiplying corresponding entries. Such a product, however, was not defined. When the "product" of two matrices is considered, the product obtained by multiplying the elements of a row by the corresponding elements of a column, and adding, is the product specified. As you recall, this product was motivated by considering parts — models applications and by considering linear transformations.

Although it was not so indicated in Chapter 1, the result obtained by multiplying corresponding entries in two column matrices of the same order, and adding the products, does have significance. This product has various names, the commonest being inner product, dot product, and scalar product. The inner product is a powerful operation that is very useful when considering perpendicularity (or orthogonality) and certain metrical questions. It is very important to speak of the "inner product" and not permit confusion to arise through slipshod use of the single word "product."

The most important fact concerning the inner product, about which students must be ever mindful, is that the inner product is a number, not a matrix or

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a vector. (In those texts where the multiplication of a matrix by a real number is called scalar multiplication, the inner product is usually called the scalar product.) Since the inner product is a number, it should not be enclosed in brackets as vectors or matrices usually are. Also, if the inner product were not a number, then such statements as

$$V \bullet W = ||V||\ ||W|| \cos \theta$$

would be invalid.

In the present text, the inner product is introduced easily through consideration of certain geometric relationships. The basis of these relationships is the law of cosines. The form in which the law of cosines is stated in the text may be less familiar than

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

which is the form usually found in trigonometry.

After the operation has been introduced through geometric considerations, the algebraic properties of the inner product are developed. Although the discussion in this chapter is limited to vectors of order two, vectors of higher order are said to be orthogonal if and only if their inner product equals zero. The idea is extremely important and can be developed at great length.

Exercises 4-5

1. (a) $\frac{5}{13}$.

   (b) $-\frac{1}{\sqrt{2}}$.

   (c) -1, parallel.

   (d) 0, orthogonal.

   (e) 0, perpendicular.

   (f) 0, orthogonal.

   (g) $\frac{20}{29}$.

   (h) 0, orthogonal.

2. $V = \begin{bmatrix} u \\ v \end{bmatrix}$, $||V|| = \sqrt{u^2 + v^2}$,

$V \bullet E_1 = u$, $V \bullet E_2 = v$,

$$\left.\begin{aligned} \frac{V \bullet E_1}{||V||} &= \frac{u}{\sqrt{u^2 + v^2}} \\ \frac{V \bullet E_2}{||V||} &= \frac{v}{\sqrt{u^2 + v^2}} \end{aligned}\right\} \quad \text{direction cosines of } V.$$

3. (a) We know (Theorem 4-5) that

$$V \bullet W = ||V|| \; ||W|| \cos \theta.$$

Consequently, the equation

$$V \bullet W = \pm \, ||V|| \; ||W||$$

is equivalent to the condition that

$$\cos \theta = \pm 1,$$

which implies that $\theta = 0$ ($\cos \theta = +1$), or else $\theta = \pi$ ($\cos \theta = -1$).

(b) The inequality $|\cos \theta| \leq 1$, together with Theorem 4-5, implies

$$(V \bullet W)^2 \leq ||V||^2 \; ||W||^2.$$

If we write this result in terms of the entries of V and W, we get

$$(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2),$$

where

$$V = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} c \\ d \end{bmatrix};$$

this simplifies to

$$0 \leq (ad - bc)^2.$$

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(c) The present inequality is obtained from (b) by taking square roots. The root of the right-hand member is not negative, though that of the left-hand member might be.

4. This follows from Theorem 4-5.

We show first that if $V \bullet W = 0$, then V and W are orthogonal. Since

$$||V||\ ||W|| \cos \theta = 0,$$

then either

(a) $||V|| = 0$,

(b) $||W|| = 0$,

(c) $\cos \theta = 0$.

If (a) or (b), then either V or W is the zero vector, which is orthogonal to any vector. If (c), then $\theta = 90^\circ$, which means V and W are perpendicular (or orthogonal).

We show next that if V and W are orthogonal then $V \bullet W = 0$.

First case: If V or W is the zero vector, $||V||\ ||W|| \cos \theta = 0$.

Second case: If neither V nor W is the zero vector, then if they are orthogonal, $\theta = 90^\circ$, whence $\cos \theta = 0$, which means

$$||V||\ ||W|| \cos \theta = 0.$$

5. (a) -20.

(b) $+2$.

(c) Nonparallel. (Another correct answer would be: "Equal in length.")

(d) -72.

(e) -3.

(f) -3.

6. (a) $V \bullet W = W \bullet V$:

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

$$-13 = -13.$$

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(b) $(rV) \bullet W = r(V \bullet W)$;

$$\left(4\begin{bmatrix}-2\\3\end{bmatrix}\right) \bullet \begin{bmatrix}2\\-3\end{bmatrix} = 4\left(\begin{bmatrix}-2\\3\end{bmatrix} \bullet \begin{bmatrix}2\\-3\end{bmatrix}\right).$$

$$\begin{bmatrix}-8\\12\end{bmatrix} \bullet \begin{bmatrix}2\\-3\end{bmatrix} = 4(-13),$$

$$-52 = -52.$$

(c) $V \bullet (W + U) = V \bullet W + V \bullet U$:

$$\begin{bmatrix}-2\\3\end{bmatrix}\left(\begin{bmatrix}2\\-3\end{bmatrix} + \begin{bmatrix}5\\1\end{bmatrix}\right) = \begin{bmatrix}-2\\3\end{bmatrix} \bullet \begin{bmatrix}2\\-3\end{bmatrix} + \begin{bmatrix}-2\\3\end{bmatrix} \bullet \begin{bmatrix}5\\1\end{bmatrix},$$

$$\begin{bmatrix}-2\\3\end{bmatrix} \bullet \begin{bmatrix}7\\-2\end{bmatrix} = -13 - 7,$$

$$-20 = -20.$$

(d) $V \bullet V \geq 0$:

$$\begin{bmatrix}-2\\3\end{bmatrix} \bullet \begin{bmatrix}-2\\3\end{bmatrix} = 13 > 0.$$

7. Let $U = \begin{bmatrix}a\\b\end{bmatrix}$, $V = \begin{bmatrix}c\\d\end{bmatrix}$, $W = \begin{bmatrix}e\\f\end{bmatrix}$.

(a) $V \bullet W = W \bullet V$:

$$\begin{bmatrix}c\\d\end{bmatrix} \bullet \begin{bmatrix}e\\f\end{bmatrix} = \begin{bmatrix}e\\f\end{bmatrix} \bullet \begin{bmatrix}c\\d\end{bmatrix},$$

$$ce + df = ec + fd,$$

$$ce + df = ce + df.$$

$(rV) \bullet W = r(V \bullet W)$:

$$\left(r\begin{bmatrix}c\\d\end{bmatrix}\right) \bullet \begin{bmatrix}e\\f\end{bmatrix} = r\left(\begin{bmatrix}c\\d\end{bmatrix} \bullet \begin{bmatrix}e\\f\end{bmatrix}\right),$$

$$\begin{bmatrix} rc \\ rd \end{bmatrix} \bullet \begin{bmatrix} e \\ f \end{bmatrix} = r(ce + df),$$

$$rce + rdf = rce + rdf.$$

$V \bullet (W + U) = V \bullet W + V \bullet U$:

$$\begin{bmatrix} c \\ d \end{bmatrix} \left(\begin{bmatrix} e \\ f \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} c \\ d \end{bmatrix} \bullet \begin{bmatrix} e \\ f \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} \bullet \begin{bmatrix} a \\ b \end{bmatrix},$$

$$\begin{bmatrix} c \\ d \end{bmatrix} \left(\begin{bmatrix} e + a \\ f + b \end{bmatrix} \right) = (ce + df) + (ca + db),$$

$$(ce + ca) + (df + db) = ce + df + ca + db.$$

$V \bullet V \geq 0$:

$$\begin{bmatrix} c \\ d \end{bmatrix} \bullet \begin{bmatrix} c \\ d \end{bmatrix} \geq 0,$$

$$c^2 + d^2 \geq 0.$$

(b) $V \bullet W = V^tW = W^tV = W \bullet V.$

$(rV) \bullet W = (rV)^tW = rV^tW = r(V \bullet W).$

$V \bullet (W + U) = V^t(W + U) = V^tW + V^tU = V \bullet W + V = U.$

$V \bullet V = V^tV \geq 0$

if $V \bullet V = 0 = V^tV$ and $V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

8. $(V + W) \bullet (V + W) = ||V + W||^2$ by Corollary 4-5-1;

$(V + W) \bullet (V + W) = V \bullet V + 2V \bullet W + W \bullet W = ||V||^2 + 2V \bullet W + ||W||^2$

by Corollary 4-5-1 and Theorem 4-6c.

9. (a) $V \bullet W = -6 + 6 = 0$, $T \bullet W = 24 - 24 = 0$,

$V \bullet T = -4 - 16 = -20 = \pm ||V||\ ||T|| = -\sqrt{5}\ \sqrt{80} = -20.$

(b) $V \bullet W = -6 + 6 = 0$, $T \bullet W = -42 + 42 = 0$,

$V \bullet T = 28 + 63 = 91 = \pm ||V||\ ||T|| = +\sqrt{13}\ \sqrt{637} = +91.$

No, T and W are not orthogonal.

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10. Let

$$V = \begin{bmatrix} u \\ v \end{bmatrix}.$$

Since V and T are parallel,

$$T = r\begin{bmatrix} u \\ v \end{bmatrix},$$

by Exercise 4–3–4. Let

$$W = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Now

$$V \bullet W = 0;$$

hence

$$au + bv = 0, \quad W \bullet T = rua + rvb = r(au + bv) = 0;$$

therefore $W \bullet T = 0$, and the vectors are orthogonal.

11. Since

$$\begin{bmatrix} r \\ s \end{bmatrix} \bullet \begin{bmatrix} -ts \\ tr \end{bmatrix} = -rts + str = 0$$

for all $r, s, t,$ the vectors are orthogonal.

12. Let

$$V = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} a \\ b \end{bmatrix},$$

where $a, b, u, v \in R$. Since V and W are orthogonal,

$$V \bullet W = au + bv = 0,$$

$$au = -bv.$$

If $u \neq 0$, then

$$a = \left(\frac{b}{u}\right) (-v).$$

Now

$$b = \left(\frac{b}{u}\right) u.$$

Let

$$t = \frac{b}{u}, \qquad\qquad t \in R.$$

Hence

$$W = \begin{bmatrix} a \\ b \end{bmatrix} = \frac{b}{u} \begin{bmatrix} -v \\ u \end{bmatrix} = t \begin{bmatrix} -v \\ u \end{bmatrix}.$$

On the other hand, if $u = 0$, then $v \neq 0$ since $V \neq \underline{0}$, so

$$b = \left(\frac{a}{-v}\right) (u), \quad a = \left(\frac{a}{-v}\right) (-v),$$

whence again

$$W = \begin{bmatrix} a \\ b \end{bmatrix} = \frac{a}{-v} \begin{bmatrix} -v \\ u \end{bmatrix} = t \begin{bmatrix} -v \\ u \end{bmatrix},$$

with

$$t = \frac{a}{-v}.$$

13\. (a) If

$$||V + W||^2 - ||V - W||^2 = 0,$$

then $V \bullet W = 0$. Let

$$V = \begin{bmatrix} a \\ b \end{bmatrix}, \quad W = \begin{bmatrix} c \\ d \end{bmatrix}.$$

so that

$$V + W = \begin{bmatrix} a + c \\ b + d \end{bmatrix}, \quad V - W = \begin{bmatrix} a - c \\ b - d \end{bmatrix},$$

Now

$$(a + c)^2 + (b + d)^2 - (a - c)^2 - (b - d)^2 = 0,$$
$$4ac + 4bd = 0,$$
$$ac + bd = 0.$$

Since $V \bullet W = ac + bd$, we have $V \bullet W = 0$.

(b) If $V \bullet W = 0$, then $||V + W||^2 - ||V - W||^2 = 0$.

Since $V \bullet W = 0$, $ac + bd = 0$.

Hence $(a + c)^2 + (b + d)^2 - (a - c)^2 - (b - d)^2 = 0$.

14. $(a^2 + b^2)(c^2 + d^2) - (ac + bd)^2$

$= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 - a^2c^2 - b^2d^2 - 2acbd = a^2d^2 + b^2c^2 - 2acbd$

$= (ad - bc)^2.$

15. We show first that if $(V + W) \bullet (V + W) = V \bullet V + W \bullet W$, then $V \bullet W = 0$.

Let $V = \begin{bmatrix} a \\ b \end{bmatrix}$, $W = \begin{bmatrix} c \\ d \end{bmatrix}$.

Since $(V + W) \bullet (V + W) = V \bullet V + W \bullet W$, we have

$$(a + c)^2 + (b + d)^2 = a^2 + b^2 + c^2 + d^2,$$
$$2ac + 2bd = 0,$$
$$ac + bd = 0; \quad \text{hence} \quad V \bullet W = 0.$$

We show next that if $V \bullet W = 0$, then $(V + W) \bullet (V + W) = V \bullet V + W \bullet W$.

Since $V \bullet W = 0$, we have $ac + bd = 0$, and $2ac + 2bd = 0$.

Therefore $(a + c)^2 + (b + d)^2 = a^2 + b^2 + c^2 + d^2$, and

$$(V + W) \bullet (V + W) = V \bullet V + W \bullet W.$$

16. Let $V = \begin{bmatrix} a \\ b \end{bmatrix}$, $W = \begin{bmatrix} c \\ d \end{bmatrix}$. Now

$$\begin{bmatrix} a + c \\ b + d \end{bmatrix} \bullet \begin{bmatrix} a - c \\ b - d \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \bullet \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} c \\ d \end{bmatrix} \bullet \begin{bmatrix} c \\ d \end{bmatrix};$$

hence

$$a^2 - c^2 + b^2 - d^2 = a^2 + b^2 - c^2 - d^2.$$

Therefore,

$$(V + W) \bullet (V - W) = V \bullet V - W \bullet W.$$

17. Let $V = \begin{bmatrix} a \\ b \end{bmatrix}$, $W = \begin{bmatrix} c \\ d \end{bmatrix}$; then

$$||V|| = \sqrt{a^2 + b^2},$$

$$||W|| = \sqrt{c^2 + d^2},$$

$$||V + W|| = \sqrt{(a + c)^2 + (b + d)^2},$$

$$= \sqrt{a^2 + 2ac + c^2 + b^2 + 2bd + d^2}.$$

Now, if we know that $A \geq 0$, $B \geq 0$ (as we do when $A = ||V + W||$, $B = ||V|| + ||W||$), then $A \leq B$ if and only if $A^2 \leq B^2$. Consequently, we may compare the squares

$$||V + W||^2$$

and

$$(||V|| + ||W||)^2.$$

Doing so, we get

$$a^2 + 2ac + c^2 + b^2 + 2bd + d^2 \leq a^2 + b^2 + 2\sqrt{a^2 + b^2}\ \sqrt{c^2 + d^2} + c^2 + d^2.$$

This is equivalent to

$$2ac + 2bd \leq 2\sqrt{a^2 + b^2}\ \sqrt{c^2 + d^2},$$

or

$$ac + bd \leq \sqrt{a^2 + b^2}\ \sqrt{c^2 + d^2}.$$

We would now like to repeat our device of "squaring both sides." Unfortunately, we can no longer be sure whether $ac + bd$ is positive, negative, or zero. Hence we cannot get an equivalent inequality. But actually it is only the "backward" implication that we care about. Fortunately, we do know that for $C \geq 0$,

$$D^2 \leq C^2$$

implies $D \leq C$. This is all we need; let

$$C = \sqrt{a^2 + b^2}\ \sqrt{c^2 + d^2}$$

and

$$D = ac + bd.$$

Then

$$D^2 \leq C^2$$

becomes

$$a^2c^2 + 2acbd + b^2d^2 \leq a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2.$$

Is this true? Well, it is equivalent to

$$2acbd \leq a^2d^2 + b^2c^2,$$

which, in turn, is equivalent to

$$0 \leq a^2d^2 - 2acdb + b^2c^2,$$

or

$$0 \leq (ad - bc)^2.$$

This last inequality must be true.

Now, trace the implication backward; since

$$0 \leq (ad - bc)^2$$

is true, the equivalent statement

$$a^2c^2 + 2acbd + b^2d^2 \leq a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$$

must be true. Consequently,

$$ac + bd \leq \sqrt{a^2 + b^2} \quad \sqrt{c^2 + d^2}$$

must also be true.

But this last statement is equivalent to the desired result.

(The logic involved in this chain of implications is probably more confusing than the actual algebraic manipulations.)

4-6. An Area and a Determinant

The idea of a determinant is reintroduced in this section, though the original identification of a determinant is given in Chapter 2. If Chapter 2 has been omitted, the impact of the present section is not diminished if the determinant is simply defined as the value $ad - bc$; that is,

$$\delta(D) = ad - bc, \quad \text{where} \quad D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

[pages 160-165]

The idea contained in this section is particularly important because it interrelates the geometrical and algebraic aspects of a vector and also interrelates vectors and matrices.

Exercises 4-6

1. (a) $\frac{|10 - 0|}{2} = 5.$

   (b) $\frac{|2 + 8|}{2} = 5.$

   (c) $\frac{|-2 - 6|}{2} = 4.$

2. (a) 5, (b) 0, (c) 2, (d) 2.5, (e) 2.

3. (a) $3969 \leq 4225$, $17.89 \leq 18$;

   (b) $100 \leq 100$, $3\sqrt{5} \leq 3\sqrt{5}$;

   (c) $100 \leq 100$, $\sqrt{5} \leq 3\sqrt{5}$.

4-7. Vector Spaces and Subspaces

In the last section of Chapter 2, we defined an algebra as a system having both the properties of a ring and of a vector space. In this section, we finally define a vector space precisely. The addition of this new concept is not going to alter any intuitive notions about an algebra that the class may possess. Rather it will provide for more abstract systems the criteria by which we can judge whether or not the systems are algebraic.

Briefly, a vector space is a type of mathematical system. The elements of the system are generally what we recognize as vectors in the present test, but they do not need to be. The definition is broad enough to encompass systems in which the elements might be linear and constant polynomials as one example, polynomials of degree two or less for a second example. It is well worth demonstrating this as a class exercise.

[pages 161-172]

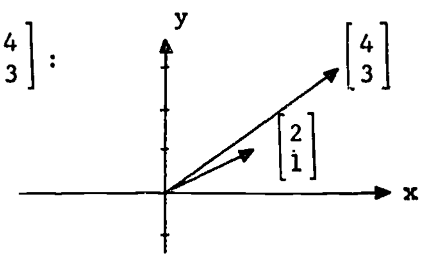
Exercises 4–7

1. (a) $\begin{bmatrix} -2 \\ 1 \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 4 \\ 3 \end{bmatrix}$,

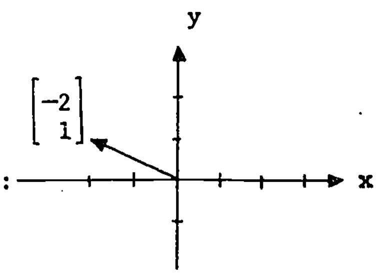
where $a = -5$, $b = 2$.

(Notice that we cannot use orthogonality here, since $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ are not orthogonal. We may, instead, use the linear equations: $2a + 4b = -2$, $a + 3b = 1$.)

The basis vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$:

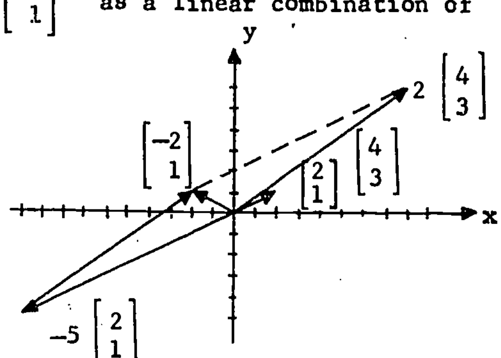


The vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$:



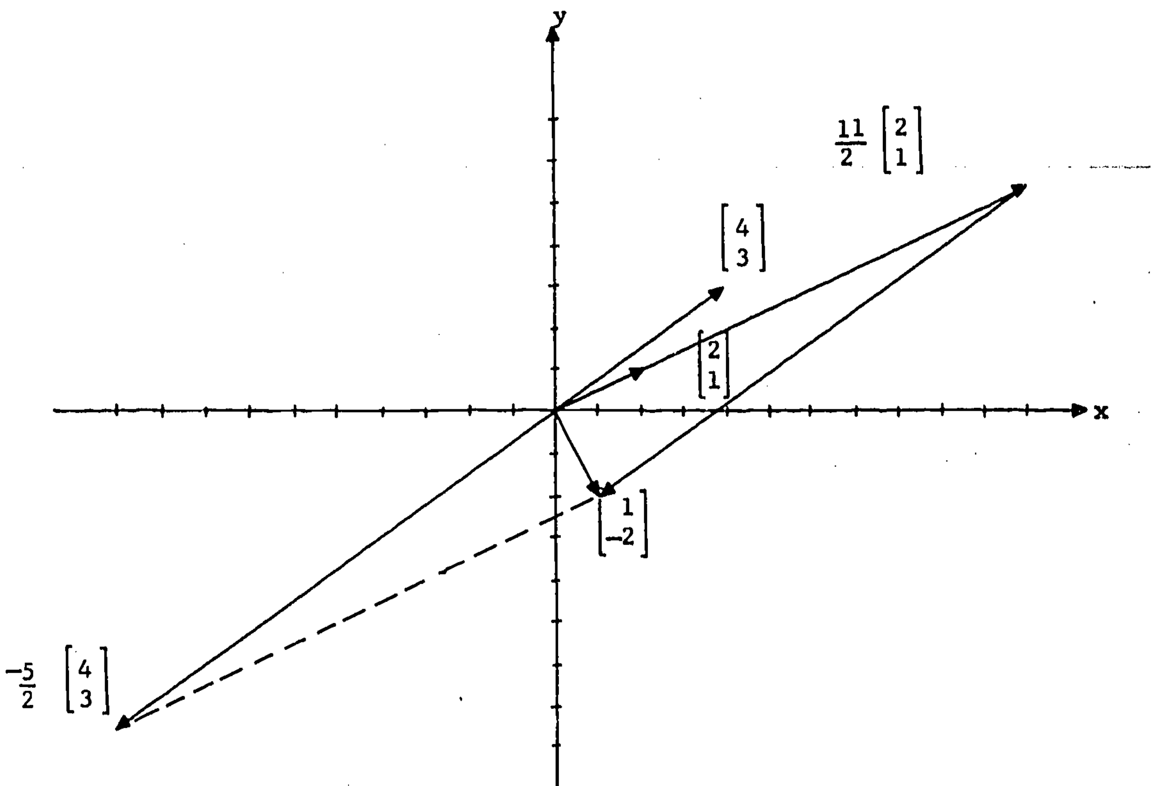
The representation of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$:

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = -5 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

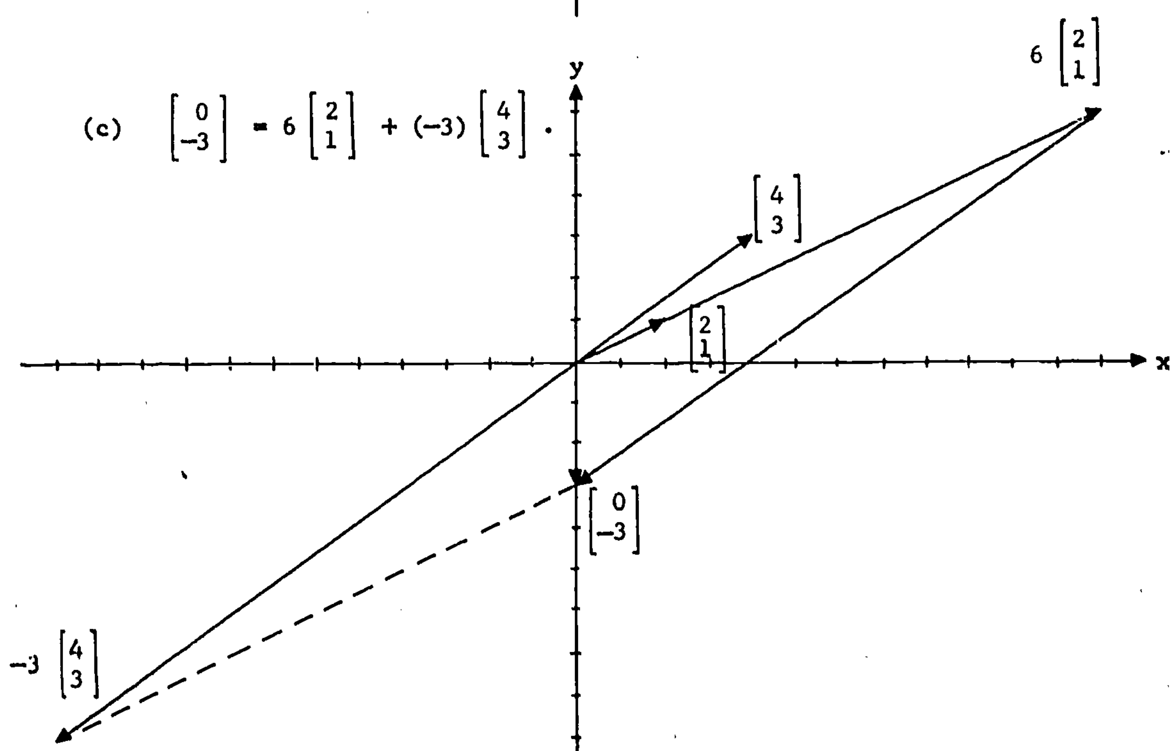


[page 173]

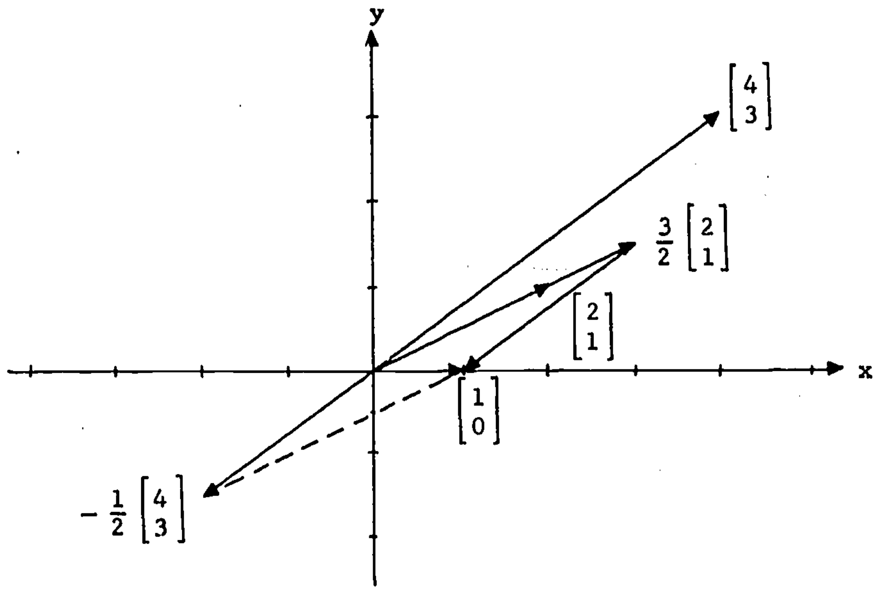
(b) $\begin{bmatrix} 1 \\ -2 \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, where $a = \frac{11}{2}$, $b = -\frac{5}{2}$.



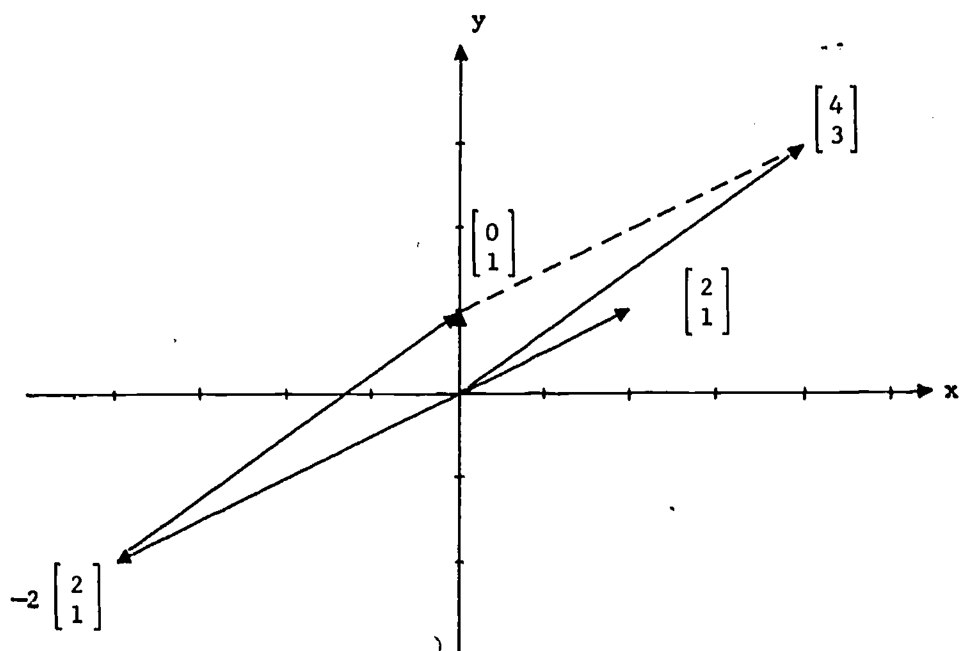
(c) $\begin{bmatrix} 0 \\ -3 \end{bmatrix} = 6 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-3) \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.



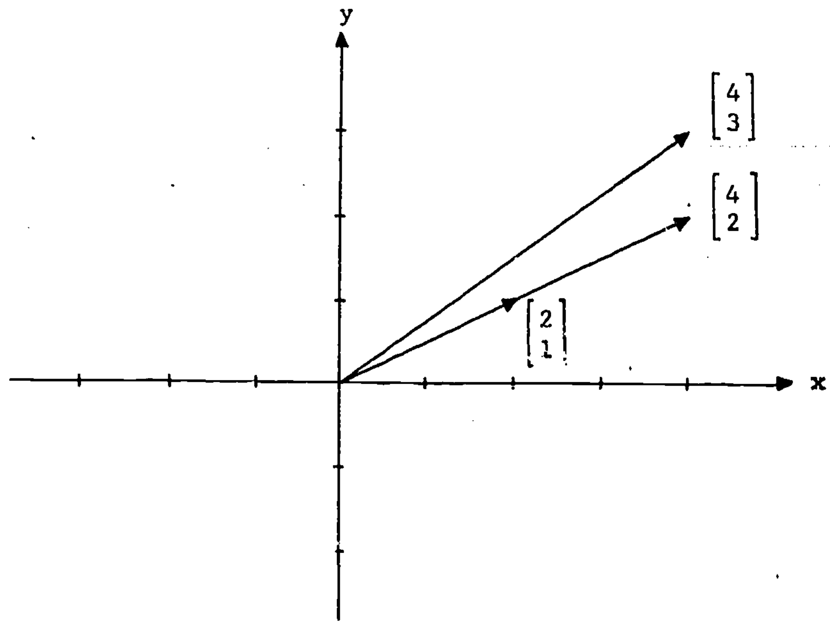
(d) $\begin{bmatrix}1\\0\end{bmatrix} = \frac{3}{2}\begin{bmatrix}2\\1\end{bmatrix} + \left(-\frac{1}{2}\right)\begin{bmatrix}4\\3\end{bmatrix}$.



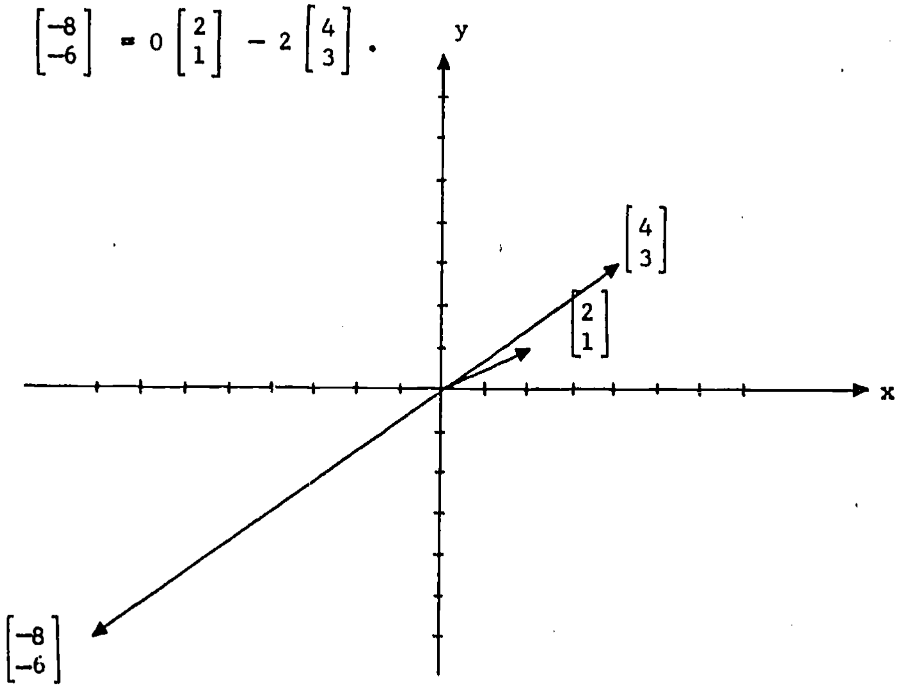
(e) $\begin{bmatrix}0\\1\end{bmatrix} = (-2)\begin{bmatrix}2\\1\end{bmatrix} + \begin{bmatrix}4\\3\end{bmatrix}$.



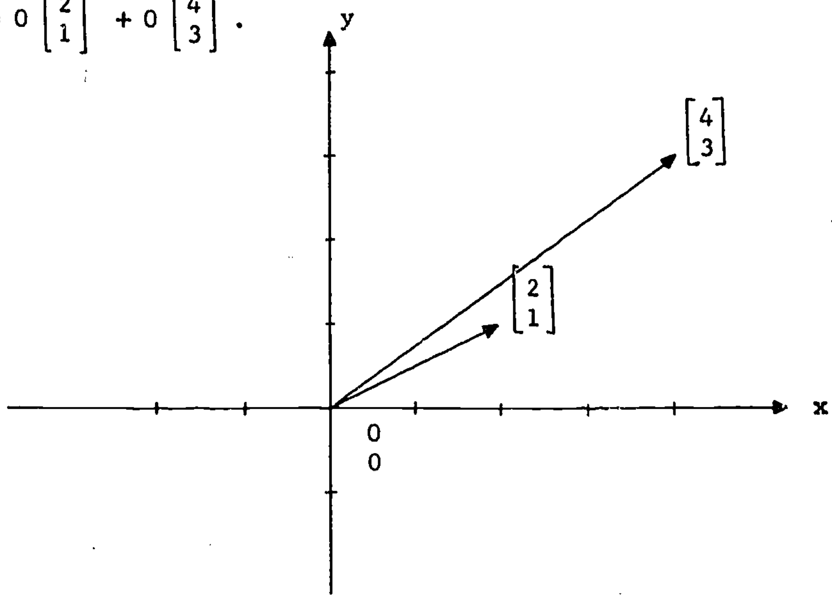
(f) $\begin{bmatrix}4\\2\end{bmatrix} = 2\begin{bmatrix}2\\1\end{bmatrix} + 0\begin{bmatrix}4\\3\end{bmatrix}$.



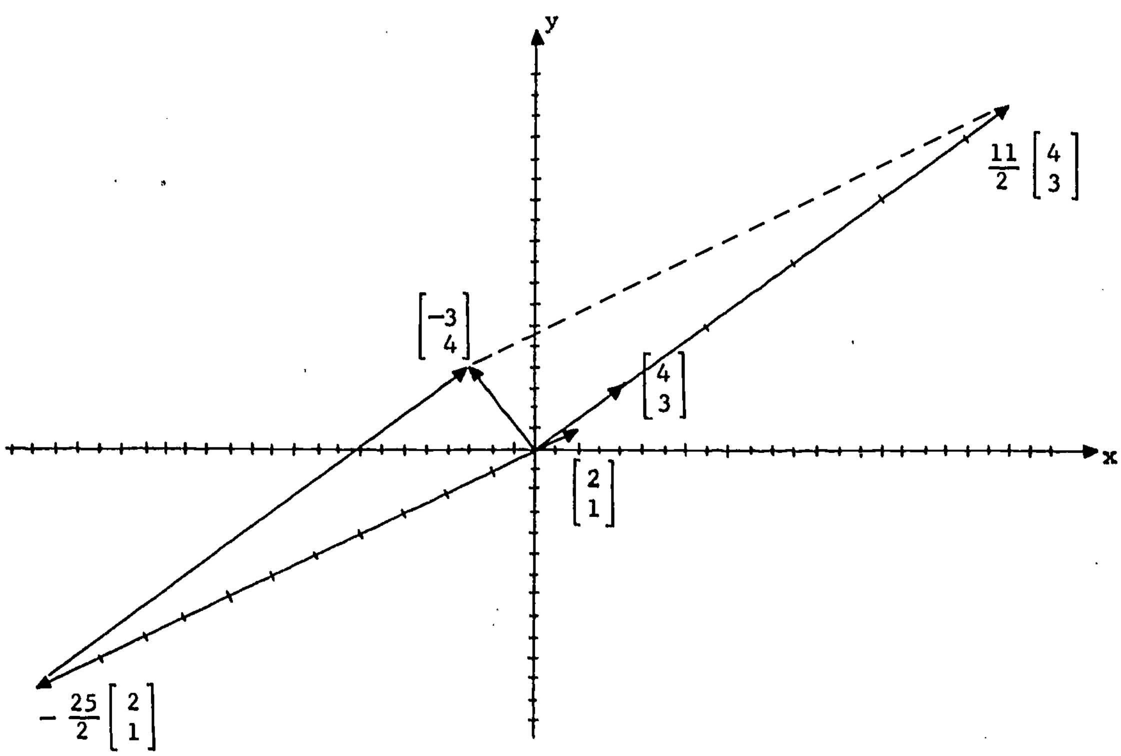
(g) $\begin{bmatrix}-8\\-6\end{bmatrix} = 0\begin{bmatrix}2\\1\end{bmatrix} - 2\begin{bmatrix}4\\3\end{bmatrix}$.



(h) $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.



(i) $\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \left(-\frac{25}{2}\right) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{11}{2} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.



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2. Notice that $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ are orthogonal. We may, if we wish, find a and b by using the inner product.

(a) $\begin{bmatrix} -2 \\ 1 \end{bmatrix} = a \begin{bmatrix} 2 \\ -1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 4 \end{bmatrix}$,

which we can abbreviate

$$Z = aV + bW.$$

Now,

$$\begin{aligned} V \bullet Z &= V \bullet (aV + bW) \\ &= a \bullet (V \bullet V) + b(V \bullet W) \\ &= 5a + 0b. \end{aligned}$$

By computing, we find $V \bullet Z = -5$. Hence, we have

$$5a = -5$$

or

$$a = -1.$$

Similarly,

$$\begin{aligned} W \bullet Z &= W \bullet (aV + bW) \\ &= a(W \bullet V) + b(W \bullet W) \\ &= 0a + 20\ b. \end{aligned}$$

Now, $W \bullet Z = 0$. Consequently, we have

$$20\ b = 0$$

or

$$b = 0.$$

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(Actually, it is possible to obtain $a = -1$, $b = 0$ by inspection in this particular problem, but we shall not pursue this point.)

The representation of Z as a linear combination of V and W is, consequently,

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 4 \end{bmatrix},$$

or, simply,

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

(b) $\begin{bmatrix} 1 \\ -2 \end{bmatrix} = a \begin{bmatrix} 2 \\ -1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 4 \end{bmatrix},$

$$a(V \bullet V) = V \bullet Z, \quad 5a = 4, \quad a = \frac{4}{5}.$$

$$b(W \bullet W) = W \bullet Z, \quad 20b = -6, \; b = -\frac{3}{10}.$$

So

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \frac{4}{5} \begin{bmatrix} 2 \\ -1 \end{bmatrix} - \frac{3}{10} \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

(c) $\begin{bmatrix} 0 \\ -3 \end{bmatrix} = \frac{3}{5} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + -\frac{3}{5} \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$

(d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{2}{5} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \frac{1}{10} \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$

(e) $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$

(f) The method we have used in parts (a)–(e) is certainly working smoothly enough — the only computation each time is to determine the two inner products $V \bullet Z$ and $W \bullet Z$, since we already know that $V \bullet V = 5$ and $W \bullet W = 20$ (and, of course, $V \bullet W = 0$, without which this method would not work so well).

[page 173]

(In Exercise 1, (a)–(i), we used a more straightforward but also more tedious method, by solving simultaneous equations. Our present "inner product" method is much more efficient, and can be done by inspection.)

Now that we have solved parts (d) and (e), we can introduce another method. This third method is really quite obvious, and, like the second method, it can be done by inspection. It will, however, require a few lines of explanation:

If we want to express $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the coordinates a and b can be determined by inspection:

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{1}$$

whence, immediately, $a = 4$ and $b = 2$.

Now,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{2}{5}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \frac{1}{10}\begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

If we substitute into equation (1), we get

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 4\left\{\frac{2}{5}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \frac{1}{10}\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\} + 2\left\{-\frac{1}{5}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\},$$

and, combining terms, we have

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \left\{4\left(\frac{2}{5}\right) + 2\left(-\frac{1}{5}\right)\right\}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \left\{4\left(\frac{1}{10}\right) + 2\left(\frac{1}{5}\right)\right\}\begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

These calculations can also be done by inspection, if we observe the patterns that are involved. After a final simplification, we have

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{6}{5}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \frac{4}{5}\begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

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As a check on our numerical work, we solve this problem again, by the "inner product" method:

$$\begin{bmatrix}4\\2\end{bmatrix} = a\begin{bmatrix}2\\-1\end{bmatrix} + b\begin{bmatrix}2\\4\end{bmatrix},$$

$$a(V \bullet V) = V \bullet Z,$$

$$5a = 6,$$

$$a = \frac{6}{5}.$$

$$b(W \bullet W) = W \bullet Z,$$

$$20\,b = 16,$$

$$b = \frac{4}{5}.$$

(g) Using the third method (i.e., using the results of parts (d) and (e)), we have

$$\begin{bmatrix}-8\\-6\end{bmatrix} = \left(-\frac{16}{5} + \frac{6}{5}\right)\begin{bmatrix}2\\-1\end{bmatrix} + \left(-\frac{8}{10} - \frac{6}{5}\right)\begin{bmatrix}2\\4\end{bmatrix}$$

$$= -2\begin{bmatrix}2\\-1\end{bmatrix} - 2\begin{bmatrix}2\\4\end{bmatrix}.$$

(h) By inspection, $a = 0$, $b = 0$.

(i) $$\begin{bmatrix}-3\\4\end{bmatrix} = -3\begin{bmatrix}1\\0\end{bmatrix} + 4\begin{bmatrix}0\\1\end{bmatrix}$$

$$= -3\left\{\frac{2}{5}\begin{bmatrix}2\\-1\end{bmatrix} + \frac{1}{10}\begin{bmatrix}2\\4\end{bmatrix}\right\} + 4\left\{-\frac{1}{5}\begin{bmatrix}2\\-1\end{bmatrix} + \frac{1}{5}\begin{bmatrix}2\\4\end{bmatrix}\right\}$$

$$= \left(-\frac{6}{5} - \frac{4}{5}\right)\begin{bmatrix}2\\-1\end{bmatrix} + \left(-\frac{3}{10} + \frac{4}{5}\right)\begin{bmatrix}2\\4\end{bmatrix}$$

$$= -2\begin{bmatrix}2\\-1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}2\\4\end{bmatrix}.$$

3. To prove that $S = \left\{r\begin{bmatrix}2\\3\end{bmatrix} \quad r \in R\right\}$ is a subspace of H, we must show that:

[page 173]

(i) $S \subset H$

(ii) [Property (a) of Definition 4-3.] If V and W are two elements of S, then $V + W$ is an element of S (i.e., the set S is closed under the operation of vector addition).

(iii) [Property (b) of Definition 4-3.] If V is an element of S, and x is any real number, then xV is an element of S (i.e., the set S is closed under the operation of multiplication by a scalar).

Here are the proofs:

(i) By the definition of H,

$$r\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}2r\\3r\end{bmatrix}$$

is an element of H. This proves part (i).

(ii) If V and W belong to S, then

$$V = s\begin{bmatrix}2\\3\end{bmatrix}$$

and

$$W = t\begin{bmatrix}2\\3\end{bmatrix},$$

for some real numbers s and t (by the definition of S).

Now, by vector addition,

$$\begin{aligned} V + W &= s\begin{bmatrix}2\\3\end{bmatrix} + t\begin{bmatrix}2\\3\end{bmatrix} \\ &= \begin{bmatrix}2s\\3s\end{bmatrix} + \begin{bmatrix}2t\\3t\end{bmatrix} \\ &= \begin{bmatrix}2s + 2t\\3s + 3t\end{bmatrix} \\ &= \begin{bmatrix}2(s + t)\\3(s + t)\end{bmatrix} \\ &= (s + t)\begin{bmatrix}2\\3\end{bmatrix}, \end{aligned}$$

which belongs to S, since $s + t$ is a real number (because the real number system is closed under addition).

This completes the proof of part (ii).

(iii) Let V be an element of S.

Then (by the definition of S),

$$V = y \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2y \\ 3y \end{bmatrix}$$

for some real number y.

If x is any real number, we must show that

$$xV$$

is an element of S.

Now

$$xV = x \begin{bmatrix} 2y \\ 3y \end{bmatrix}$$

$$= \begin{bmatrix} 2xy \\ 3xy \end{bmatrix}$$

$$= xy \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

which does belong to S since xy is a real number (because the real number system is closed under multiplication). This completes the proof.

4. This proof is virtually identical with that for Exercise 3, except that abstract notation must be used instead of explicit computation. We shall proceed as in Exercise 4:

(i) rW belongs to H, since H is a vector space, and is therefore closed under multiplication by a scalar.

(ii) If V, U belong to $S = \{rW \mid r \in R\}$, then $V = sW$ and $U = tW$, where $s \in R$, $t \in R$ (by the definition of S).

$$U + V = tW + sW$$

$$= (t + s)W$$

by Property IIc of Theorem 4-2, p. 134 and $(t + s)W$ belongs to S since the real number system is closed under addition, and hence $t + s$ is a real number (use the definition of S).

(iii) If $V \in S$, then $V = sW$, where $s \in R$ (by the definition of S). But, for tV, where $t \in R$, we have

$$\begin{aligned} tV &= t(sW) \\ &= (ts)W \end{aligned}$$

(by Property IIb of Theorem 4-2, p. 134 of the text.)

Now, since the real number system is closed under multiplication, we know that ts is a real number. Therefore tV is a real number times W. Therefore, by the definition of S,

$$tV \in S.$$

This completes the proof.

5. (a) Yes, it is a subspace.

(b) No, because rV, where $r \in R$, will not necessarily be of this form. (If we used the set of integers as our scalars, then in that case the answer here would be "Yes.")

(c) No, since rV, where $r \in R$, would not necessarily be of this form. (If we used the system of rational numbers as our scalars, the answer would be "Yes.")

(d) $$\begin{aligned} &(2u_1 - v_1) + (2u_2 - v_2) \\ &\quad = 2(u_1 + u_2) - (v_1 + v_2). \end{aligned}$$

Consequently, this is a subspace.

(e) No, this is not a subspace. If V and W belong to this set, then the sum of the entries in $V + W$ will be 4, instead of 2.

(f) No, since this set is not closed under addition. (For example, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ satisfy $uv = 0$, but their sum $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ does not.)

6. To prove that $U + W$ belongs to F, observe that $U + W$ is a linear combination of U and W. Similarly, sU is a linear combination of U and W.

7. Suppose

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = a \begin{bmatrix} -2 \\ 5 \end{bmatrix} + b \begin{bmatrix} 6 \\ -15 \end{bmatrix}.$$

Then we have

$$\begin{aligned} -2a + 6b &= 3, \\ 5a - 15b &= 1. \end{aligned}$$

Multiplying the first of these equations by $-5/2$ produces

$$5a - 15b = -\frac{15}{2},$$

which contradicts the second equation of the system, so there can be no solution for a and b.

As a second solution to this problem, we can observe that

$$\begin{bmatrix} 6 \\ -15 \end{bmatrix} = -3 \begin{bmatrix} -2 \\ 5 \end{bmatrix},$$

so that

$$V \equiv \begin{bmatrix} -2 \\ 5 \end{bmatrix} \quad \text{and} \quad U \equiv \begin{bmatrix} 6 \\ -15 \end{bmatrix}$$

are collinear. Now, if $Z \equiv \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ could be expressed as a linear combination of V and U, we would have

$$\begin{aligned} Z &= aV + bU \\ &= aV + (-3b)V \\ &= (a - 3b)V, \end{aligned}$$

so that Z would be collinear with V. However, $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is not collinear

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with $\begin{bmatrix}-2\\5\end{bmatrix}$, and this contradiction implies that Z cannot be a linear combination of U and V.

8. If

$$Z = aU + bV,$$

and

$$U = cV,$$

then

$$Z = acV + bV$$
$$= (ac + b)V,$$

so the result is the set of all vectors collinear with V.

9. Let $V \in F$, i.e., $V \in F_1$ and also $V \in F_2$. We must show that $rV \in F$. Now $rV \in F_1$, since F_1 is a vector space. Similarly, $rV \in F_2$. Consequently, rV belongs to both F_1 and F_2; by the definition of F, therefore, rV belongs to F.

The proof that $V + W \in F$ is quite similar.

10. We can replace the statement:

> If V and W are not collinear, then any vector Z of H can be expressed as a linear combination
>
> $$Z = aV + bW.$$

by the equivalent statement:

> If there exists a vector Z that cannot be expressed as a linear combination
>
> $$Z = aV + bW,$$
>
> then V and W are collinear.

We now prove this second statement. The vector equation

[pages 173-174]

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = a \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + b \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

is equivalent to the system of equations

$$v_1\, a + w_1\, b = z_1,$$

$$v_2\, a + w_2\, b = z_2,$$

where the "unknowns" are a and b.

Now we are supposing that this system of equations is impossible, i.e., there are no solutions a and b. But there are no solutions a and b if and only if the determinant of the coefficients

$$\begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix}$$

is zero.

Now, by Theorem 4-7 (p. 163) this determinant is zero if and only if the parallelogram determined by $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ has zero area, i.e., if and only if V and W are collinear.

This completes the proof.

To prove the converse, we need to observe merely that every implication in the preceding proof is in fact an equivalence, so that the two conditions are equivalent: Each implies the other.

11. Suppose the statement were false. Then there would exist a vector Z for which there would be at least two different representations:

$$Z = a_1 V + b_1 W = a_2 V + b_2 W,$$

where

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \neq \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}.$$

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By subtraction,

$$(a_1 - a_2)V + (b_1 - b_2)W = 0,$$

where either $a_1 - a_2 \neq 0$, or $b_1 - b_2 \neq 0$, or both.

Suppose that $a_1 - a_2 \neq 0$ (or else relabel). Then we can divide, obtaining

$$V = -\frac{b_1 - b_2}{a_1 - a_2} W ,$$

so that V and W would be collinear.

This completes the proof.

12\. Let

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = x \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + z \begin{bmatrix} -2 \\ -4 \\ 5 \end{bmatrix}.$$

This is equivalent to the following system:

$$\begin{aligned} 3x + 2y - 2z &= u, \\ 2x - y - 4z &= v, \\ -x + y + 5z &= w, \end{aligned}$$

and this system has a unique solution; see Exercise 5(a) of Section 3-3 on page 118 of the text and page 87 of this Commentary. Namely, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{17} & \frac{12}{17} & \frac{10}{17} \\ \frac{6}{17} & -\frac{13}{17} & -\frac{8}{17} \\ -\frac{1}{17} & \frac{5}{17} & \frac{5}{17} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

For example, if for

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad \text{we take} \quad \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix},$$

we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{17} & \frac{12}{17} & \frac{10}{17} \\ \frac{6}{17} & -\frac{13}{17} & -\frac{8}{17} \\ -\frac{1}{17} & \frac{5}{17} & \frac{7}{17} \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix},$$

so that

$$\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \\ 5 \end{bmatrix}.$$

Chapter 5
TRANSFORMATIONS OF THE PLANE

5-1. Functions and Geometric Transformations

A geometric point of view is a very valuable asset in mathematics, for skill in geometric visualization can often lend insight and obviate much tedious calculation. The process of acquiring a "geometric point of view" — i.e., of developing one's ability to create and use appropriate geometric models — often requires real effort, but the mathematical "payoff" is well worth it.

Exercises 5-1

1. (a) $\begin{bmatrix} 15 \\ 3 \end{bmatrix}$.

   (b) $\begin{bmatrix} 6 \\ -3 \end{bmatrix}$.

   (c) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

   (d) $\begin{bmatrix} 21 \\ -9 \end{bmatrix}$.

   (e) $3\begin{bmatrix} 12 \\ -2 \end{bmatrix} = \begin{bmatrix} 36 \\ -6 \end{bmatrix}$.

   (f) $3\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \end{bmatrix}$.

2. (a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

   (b) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

   (c) $\begin{bmatrix} -3 \\ 0 \end{bmatrix}$.

   (d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

   (e) $\begin{bmatrix} -15 \\ 0 \end{bmatrix}$.

   (f) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

3. (a) No change; this is the "identity" mapping.

   (b) Collapses the plane to a single point, namely, the origin.

   (c) Expands the plane uniformly by a factor a, if $a > 1$ (i.e., a distortionless magnification).

   If $a = 1$, this is again the identity mapping.

   If $0 < a < 1$, this is a uniform shrinking of the entire plane.

   (d) This is the same as part (c), followed by a reflection in the origin.

   (e) Projects every point in the plane perpendicularly onto the y axis.

(f) Maps the plane onto the line $y = x$, by translating every point parallel to the x axis.

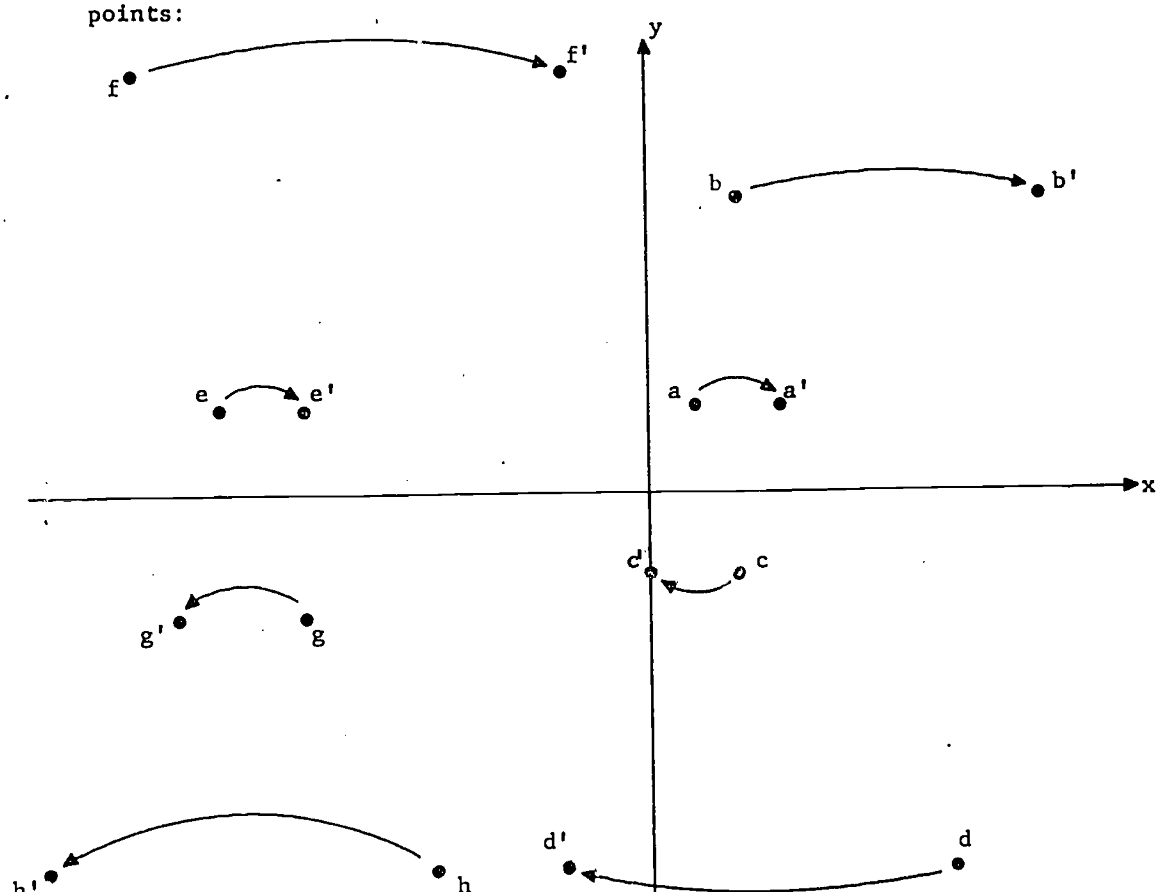
(g) A reflection in the y axis.

(h) A reflection in the x axis.

(i) Every point is moved parallel to the x axis, so that its distance from the y axis is doubled. (You might visualize the points of the plane as molecules in a gas. The motion corresponds to suddenly creating a vacuum at the far left and far right, at "$x = \infty$" and at "$x = -\infty$.")

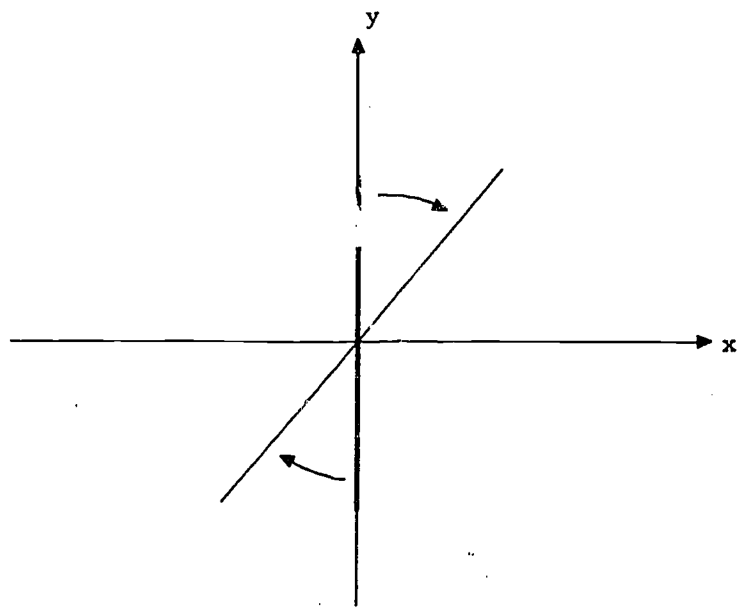
(j) A uniform magnification by a factor of 3.

(k) A "shearing" motion, in which every point moves parallel to the x axis. Points above the x axis move to the right, and points twice as far from the x axis are moved twice as great a distance. Points below the x axis move to the left, and points twice as far from the x axis are translated through twice as great a distance. We illustrate with 8 points:

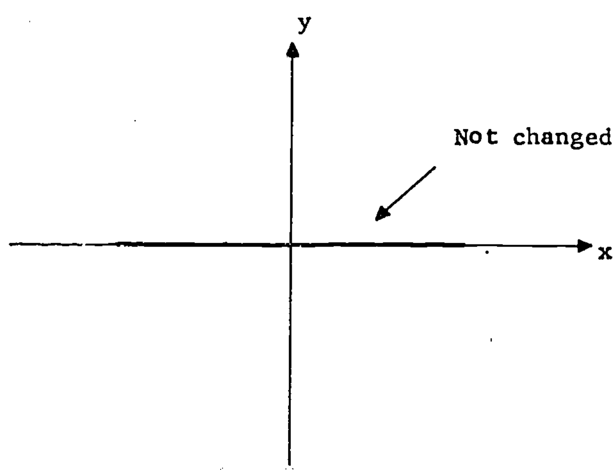


Another way to describe this is as follows:

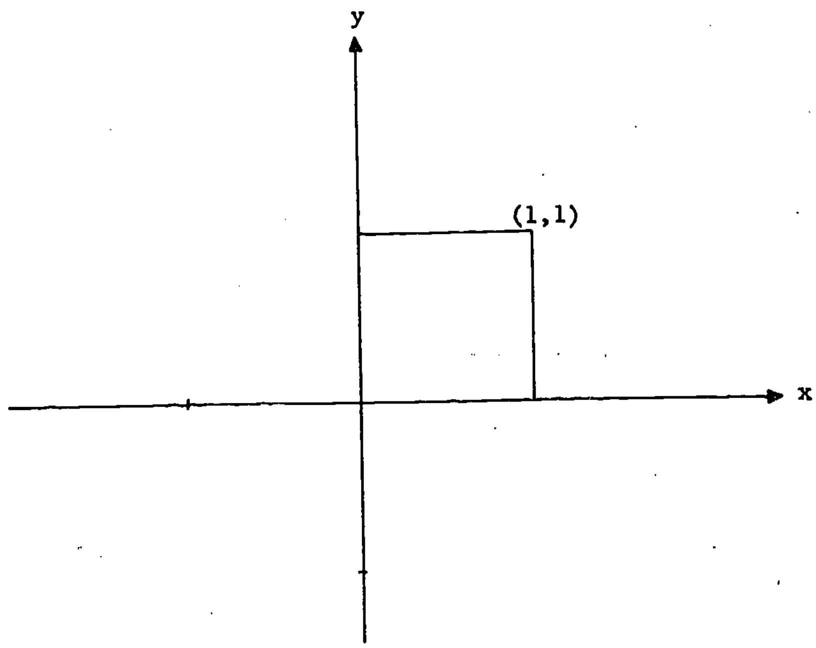
The line $x = 0$ is transformed into the line $y = x$:



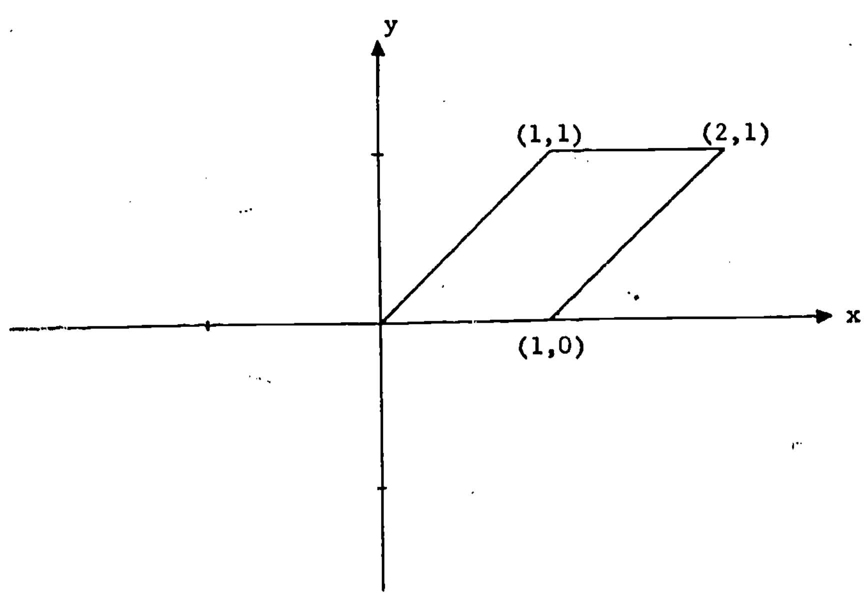
The line $y = 0$ remains invariant:



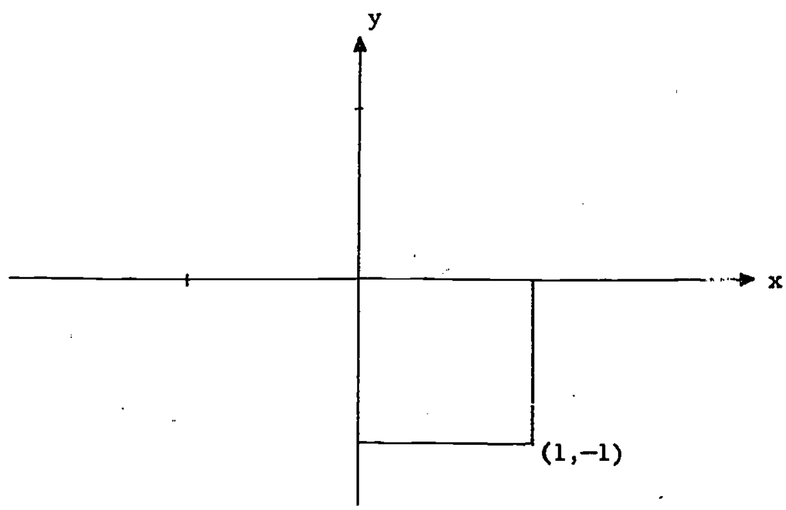
The square $0 \leq x \leq 1$, $0 \leq y \leq 1$,



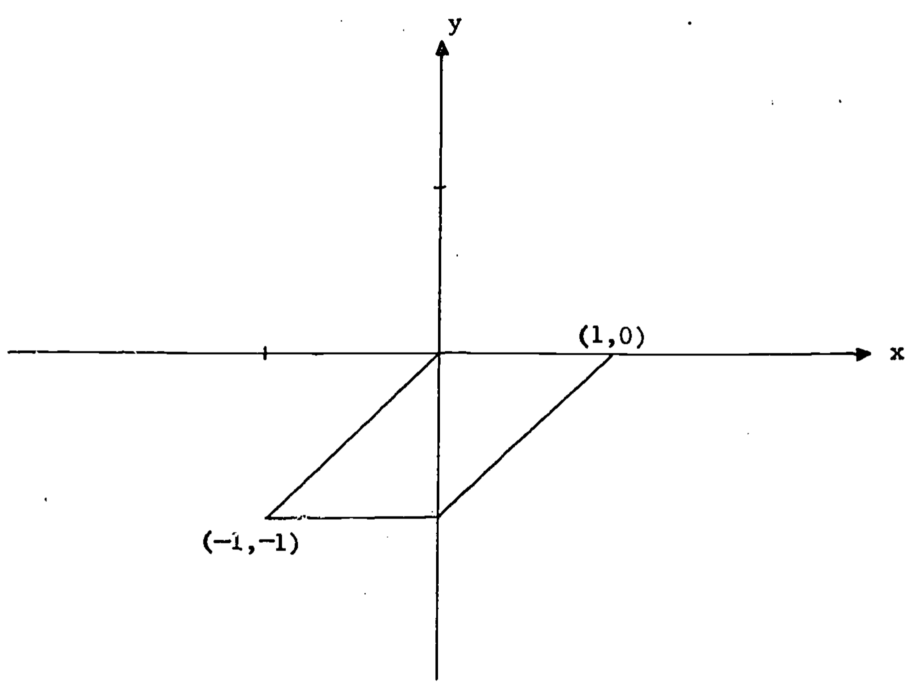
is transformed into a parallelogram:



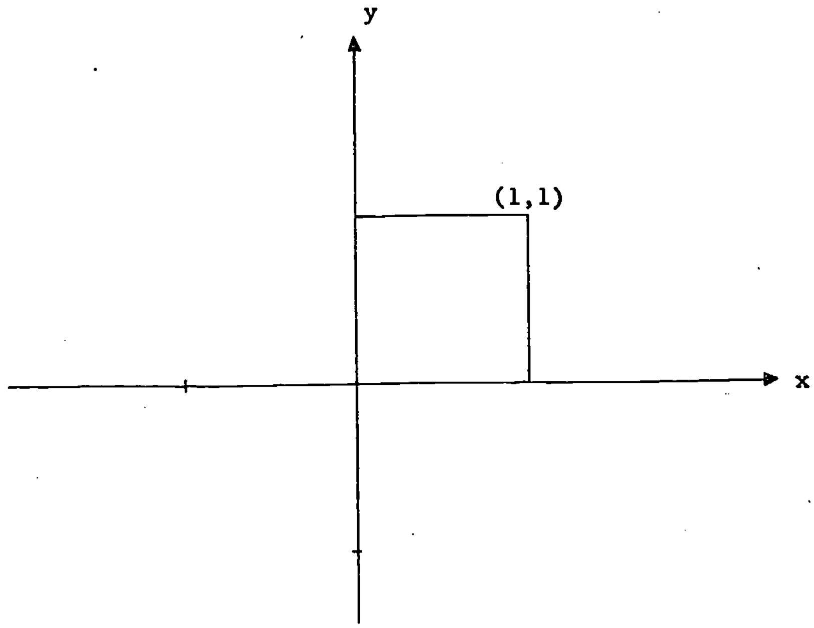
The square $0 \leq x \leq 1$, $-1 \leq y \leq 0$,



is mapped into the parallelogram:

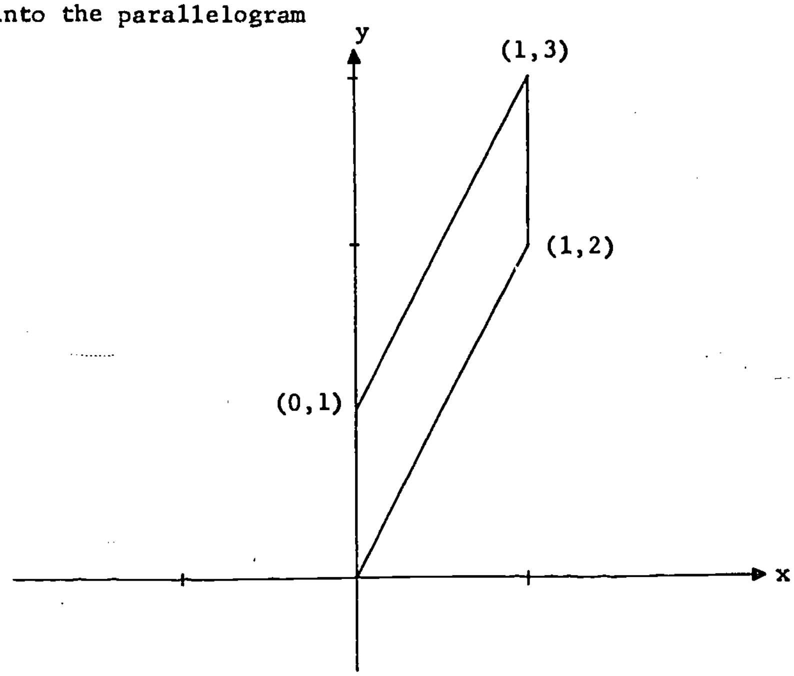


(1) This is another "shearing" motion. Each point moves parallel ... the y axis. The square



$$0 \le x \le 1, \qquad 0 \le y \le 1,$$

is mapped into the parallelogram



$$0 \le x \le 1, \qquad 2x \le y \le 2x + 1.$$

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(m) Another shearing motion. Each point moves parallel to the x axis.

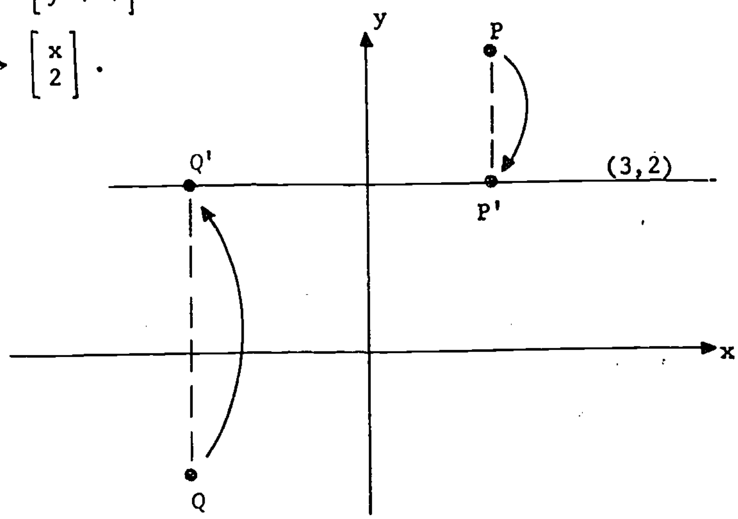
(n) Still another shearing motion. Each point moves parallel to the y axis.

4. These are one-to-one: (a), (c), (d), (g), (h), (i), (j), (k), (l), (m), (n). Transformation (b) carries the original 2-dimensional space into a point (which we call a 0-dimensional space); transformations (e) and (f) map the original space into a line (a 1-dimensional space).

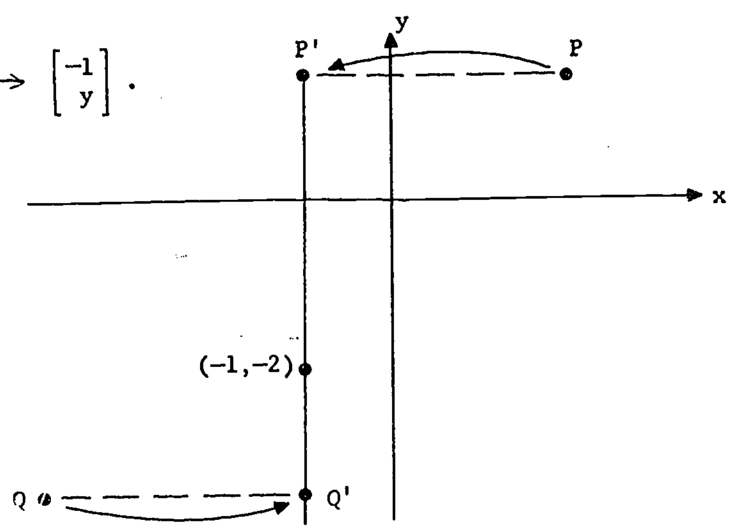
5. Let $V = \begin{bmatrix} x \\ y \end{bmatrix}$.

(a) $V \longrightarrow \begin{bmatrix} x + 1 \\ y + 4 \end{bmatrix}$.

(b) $V \longrightarrow \begin{bmatrix} x \\ 2 \end{bmatrix}$.

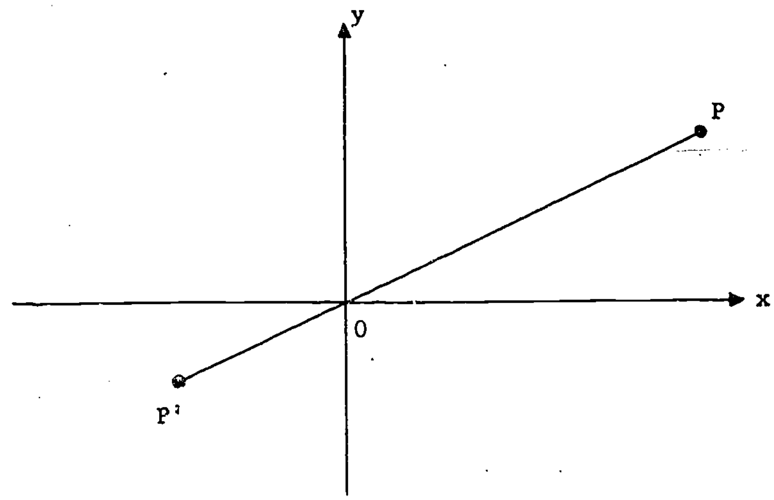


(c) $V \longrightarrow \begin{bmatrix} -1 \\ y \end{bmatrix}$.



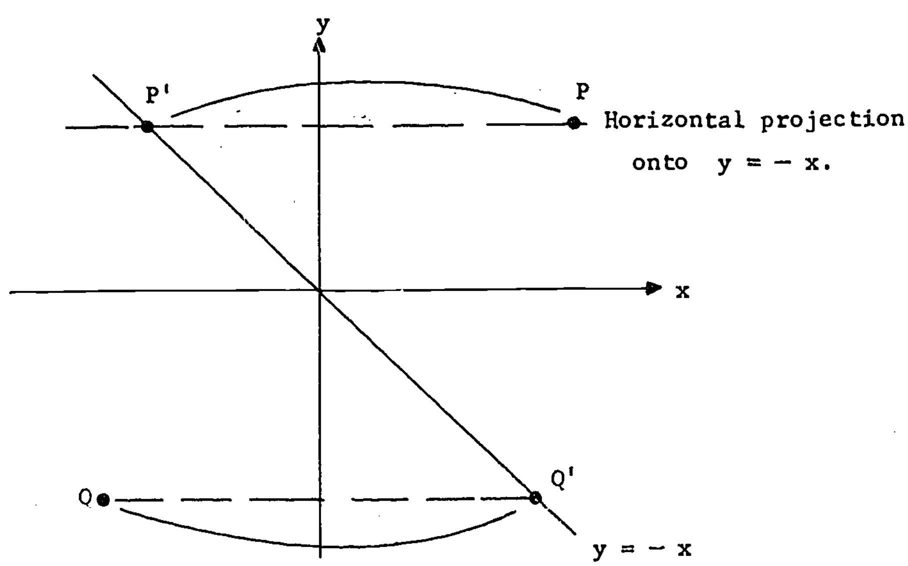
[pages 187, 188]

(d) $V \longrightarrow \begin{bmatrix} -\frac{x}{2} \\ -\frac{y}{2} \end{bmatrix}$.

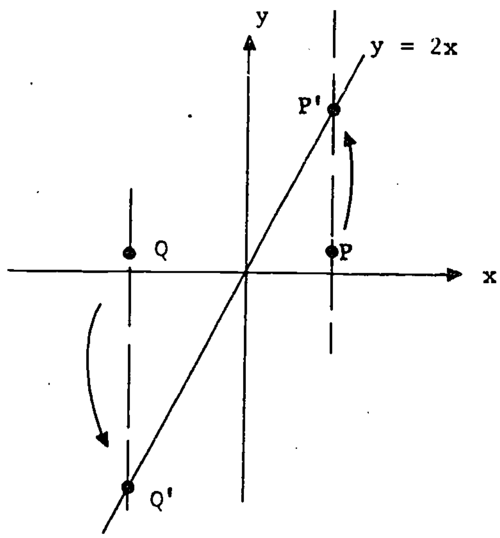


$||OP'|| = \frac{1}{2}\ ||OP||$

(e) $V \longrightarrow \begin{bmatrix} -y \\ y \end{bmatrix}$.



(f) $V \longrightarrow \begin{bmatrix} x \\ 2x \end{bmatrix}$



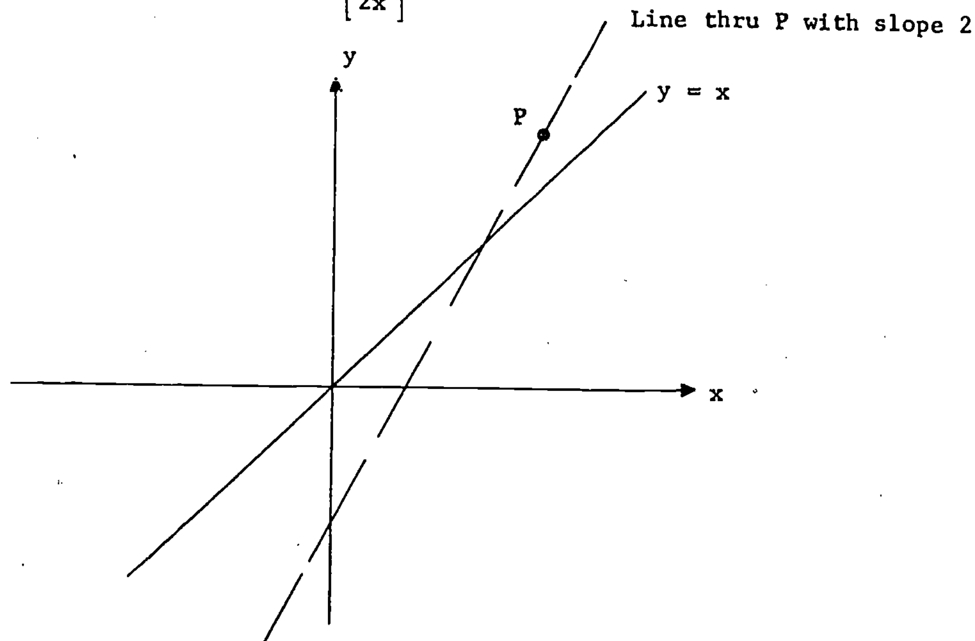
Since we know that the x coordinate is not changed, we have

$$V \longrightarrow \begin{bmatrix} x \\ ? \end{bmatrix}.$$

Now, we know the image P' (or Q') lies on the line $y = 2x$. Hence, the second entry in the vector must be $2x$, and we have

$$V \longrightarrow \begin{bmatrix} x \\ 2x \end{bmatrix}.$$

6.



Let P have coordinates (p_1, p_2). Then the line through P with slope 2 is

$$y - p_2 = 2(x - p_1).$$

This line intersects the line $y = x$ at the point (x,y) found by solving the system

$$y - p_2 = 2(x - p_1),$$
$$y = x.$$

An equivalent system,

$$x - p_2 = 2(x - p_1),$$
$$y = x,$$

can be solved to get

$$x = 2p_1 - p_2,$$
$$y = 2p_1 - p_2,$$

so the coordinates of P' are

$$\begin{bmatrix} 2p_1 - p_2 \\ 2p_1 - p_2 \end{bmatrix},$$

i.e., the vector transformation can be written

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2p_1 - p_2 \\ 2p_1 - p_2 \end{bmatrix},$$

or, in the x, y notation,

$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} 2x - y \\ 2x - y \end{bmatrix}.$$

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7. (a) This is merely matrix multiplication:

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 4x + 3y \end{bmatrix}.$$

(b) $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$

(c) This is the subspace of vectors collinear with $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

8. (a) $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}.$

(b) $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 21 \end{bmatrix}.$

(c) $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

(d) $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}.$

9. (a) First method:

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 22 \end{bmatrix}.$$

Second method (using linearity):

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$$

and multiplication by 2 yields

$$4\begin{bmatrix} 4 \\ 11 \end{bmatrix} = \begin{bmatrix} 8 \\ 22 \end{bmatrix}.$$

(b) First method:

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 13 \\ 32 \end{bmatrix}.$$

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Second method (using linearity):

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 21 \end{bmatrix},$$

$$\begin{bmatrix} 9 \\ 21 \end{bmatrix} + \begin{bmatrix} 4 \\ 11 \end{bmatrix} = \begin{bmatrix} 13 \\ 32 \end{bmatrix}.$$

(c) First method:

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 18 \end{bmatrix}.$$

Second method (using linearity):

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix};$$

$$\begin{bmatrix} 4 \\ 11 \end{bmatrix} + \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ 18 \end{bmatrix}.$$

(d) First method:

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 14 \\ 31 \end{bmatrix}.$$

Second method (using linearity):

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 16 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \end{bmatrix},$$

$$\begin{bmatrix} 4 \\ 16 \end{bmatrix} + \begin{bmatrix} 10 \\ 15 \end{bmatrix} = \begin{bmatrix} 14 \\ 31 \end{bmatrix}.$$

(e) First method:

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 17 \end{bmatrix}.$$

Second method (using linearity):

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix},$$

$$\begin{bmatrix} 2 \\ 8 \end{bmatrix} + \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 8 \\ 17 \end{bmatrix}.$$

(f) First method:

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 29 \end{bmatrix}.$$

Second method (using linearity):

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 13 \\ 32 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix},$$

$$\begin{bmatrix} 13 \\ 32 \end{bmatrix} + \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 11 \\ 29 \end{bmatrix}.$$

10. (a) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix},$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}.$$

Now, dividing by $\sqrt{2}$, we get

$$(5, 1) \longrightarrow \left(\frac{6}{\sqrt{2}}, \frac{-4}{\sqrt{2}}\right),$$

$$(-1, -3) \longrightarrow \left(\frac{-1}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right).$$

The distance from $(5, 1)$ to $(1, -2)$ is

$$\sqrt{4^2 + 3^2} = 5.$$

The distance from $\left(\frac{6}{\sqrt{2}}, \frac{-4}{\sqrt{2}}\right)$ to $\left(\frac{-1}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right)$ is

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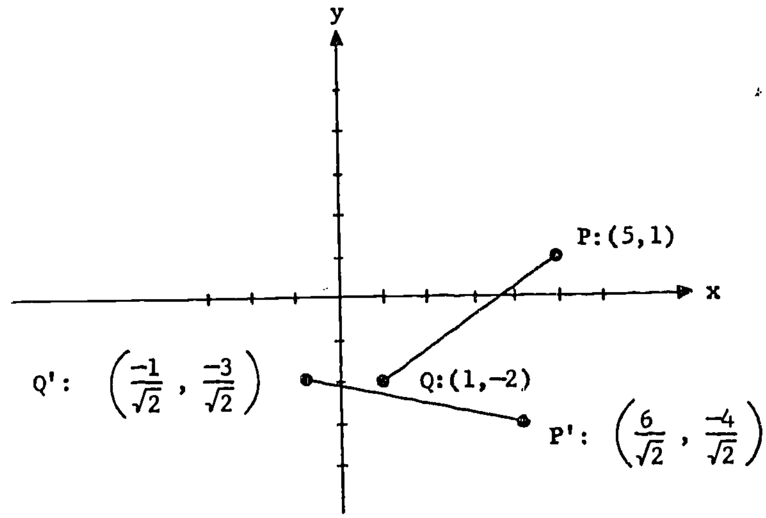
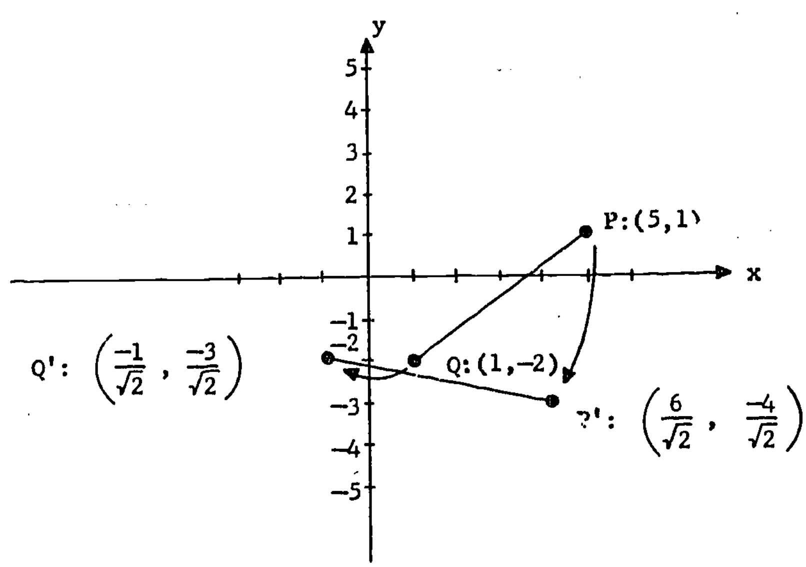
$$\frac{1}{\sqrt{2}} \sqrt{7^2 + 1^2} = \sqrt{\frac{50}{2}} = 5.$$



Actually, the transformation is a clockwise rotation of the plane through an angle of 45°:



[page 189]

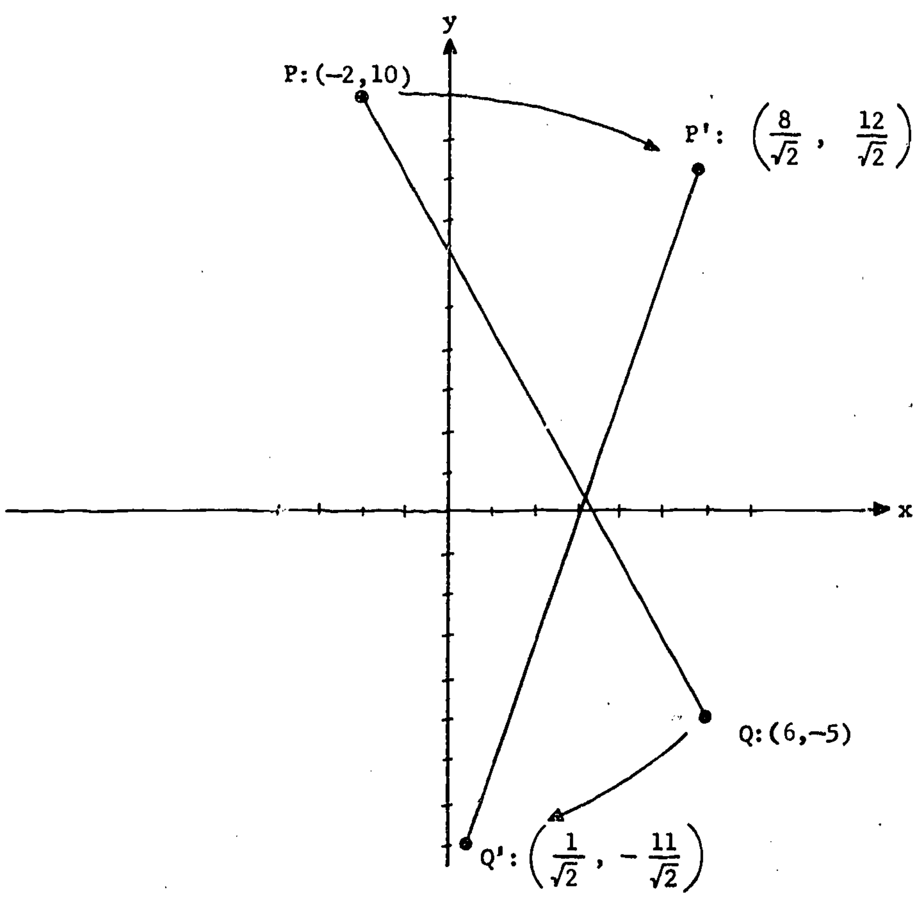
(b) $$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 10 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -11 \end{bmatrix}.$$

Dividing by $\sqrt{2}$, we find

$$\begin{bmatrix} -2 \\ 10 \end{bmatrix} \longrightarrow \begin{bmatrix} \frac{8}{\sqrt{2}} \\ \frac{12}{\sqrt{2}} \end{bmatrix}, \qquad P \longrightarrow P',$$

$$\begin{bmatrix} 6 \\ -5 \end{bmatrix} \longrightarrow \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{11}{\sqrt{2}} \end{bmatrix}, \qquad Q \longrightarrow Q'.$$



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(c)
$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ -a+b \end{bmatrix},$$
$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} c+d \\ -c+d \end{bmatrix}.$$

Dividing by $\sqrt{2}$, we get

$$\begin{bmatrix} a \\ b \end{bmatrix} \longrightarrow \begin{bmatrix} \frac{a+b}{\sqrt{2}} \\ \frac{-a+b}{\sqrt{2}} \end{bmatrix},$$

$$\begin{bmatrix} c \\ d \end{bmatrix} \longrightarrow \begin{bmatrix} \frac{c+d}{\sqrt{2}} \\ \frac{-c+d}{\sqrt{2}} \end{bmatrix}.$$

Now, the distance from (a, b) to (c, d) is

$$\sqrt{(a-c)^2 + (b-d)^2},$$

and the distance from $\left(\frac{a+b}{\sqrt{2}}, \frac{-a+b}{\sqrt{2}}\right)$ to $\left(\frac{c+d}{\sqrt{2}}, \frac{-c+d}{\sqrt{2}}\right)$ is

$$\sqrt{\frac{(a+b-c-d)^2 + (b-a+c-d)^2}{2}}$$

$$= \sqrt{\frac{(a-c)^2 + 2(a-c)(b-d) + (b-d)^2 + (b-d)^2 + 2(b-d)(c-a) + (a-c)^2}{2}}$$

$$= \sqrt{\frac{(a-c)^2 + (b-d)^2 + (b-d)^2 + (a-c)^2}{2}}$$

$$= \sqrt{(a-c)^2 + (b-d)^2}.$$

Since the distance between any two points is preserved, the transformation must be a rigid motion. As a matter of fact, it is actually a rotation clockwise through an angle of 45°, as we remarked earlier, but

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it may not be easy just yet to see why this is true. Your brightest students may well be able to prove this fact even at this stage of the game.

5-2. Matrix Transformations

For this section, the student text is largely self-explanatory. Such additional remarks as we wish to present are intermingled with the solutions to the problems, since the problems themselves give meaning to the remarks.

Exercises 5-2

1\. (a) (i) $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$,

(ii) $y = 2x$.

(b) (i) $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ -2 \end{bmatrix} = \begin{bmatrix} -7 \\ -18 \end{bmatrix}$,

(ii) $y = \frac{18}{7}x$.

(c) (i) $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$,

(ii) $y = \frac{6}{4}x = \frac{3}{2}x$.

(d) (i) $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \end{bmatrix}$,

(ii) $y = \frac{17}{3}x$.

(e) (i) $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \end{bmatrix}$,

(ii) $y = 4x$.

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(f) (i) $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} -6 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix}$,

(ii) $y = 6x$.

2. This problem uses the linearity property

$$T(aV + bW) = aTV + bTW,$$

and the fact that V and W, if not collinear, form a basis H. Hence, any vector Z can be written as a linear combination

$$Z = sV + tW.$$

If, now, we know TV and TW, we can compute TZ as

$$\begin{aligned} TZ &= T(sV + tW) \\ &= sTV + tTW. \end{aligned}$$

(a) Evidently, we want to use $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (which are <u>not</u> collinear!) as a basis. The first problem, then, is to find a and b to give us a representation of $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, i.e.,

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = a\begin{bmatrix} 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

In this case, $a = 1$, $b = 1$.

Consequently,

$$\begin{aligned} T\begin{bmatrix} 3 \\ 3 \end{bmatrix} &= T\begin{bmatrix} 2 \\ 1 \end{bmatrix} + T\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 9 \\ 6 \end{bmatrix}. \end{aligned}$$

(b) $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = a\begin{bmatrix} 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \end{bmatrix}$,

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2a + b \\ a + 2b \end{bmatrix},$$

$$2a + b = 4, \quad a + 2b = 2,$$

$$a = 2, \quad b = 0,$$

$$T\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\,T\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 2 \end{bmatrix}.$$

(c) First, we represent $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as a linear combination of the basis vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. (The idea, of course, is that we know the fate of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ under the transformation T.)

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = a\begin{bmatrix} 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$2a + b = 1, \quad a + 2b = -1,$$

$$a = 1, \quad b = -1.$$

Thus,

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Now, because of the <u>linearity</u> of the transformation T, we know that

$$T\begin{bmatrix} 1 \\ -1 \end{bmatrix} = T\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-1)\,T\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

(d) $\begin{bmatrix} -2 \\ 2 \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 2 \\ 1 \end{bmatrix}$,

$$a = -2, \quad b = 2,$$

$$T\begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2T\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2T\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= -2\begin{bmatrix} 5 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -10 \\ -2 \end{bmatrix} + \begin{bmatrix} 8 \\ 10 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

(e) $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = a\begin{bmatrix} 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \end{bmatrix}$,

$$a = \frac{1}{3}, \quad b = \frac{1}{3},$$

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{3}T\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3}T\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 5 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{3} \\ \frac{1}{3} \end{bmatrix} + \begin{bmatrix} \frac{4}{3} \\ \frac{5}{3} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

(f) $\begin{bmatrix} 4 \\ 5 \end{bmatrix} = a\begin{bmatrix} 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \end{bmatrix}$,

$$a = 1, \quad b = 2,$$

$$T\begin{bmatrix} 4 \\ 5 \end{bmatrix} = T\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2T\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 13 \\ 11 \end{bmatrix}.$$

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Alternative method (using the answer to part (a)):

$$T\begin{bmatrix}4\\5\end{bmatrix} = T\begin{bmatrix}3\\3\end{bmatrix} + T\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}9\\6\end{bmatrix} + \begin{bmatrix}4\\5\end{bmatrix} = \begin{bmatrix}13\\11\end{bmatrix}.$$

(g) $\begin{bmatrix}2\\3\end{bmatrix} = a\begin{bmatrix}2\\1\end{bmatrix} + b\begin{bmatrix}1\\2\end{bmatrix}$

$$a = \frac{1}{3}, \quad b = \frac{4}{3},$$

$$T\begin{bmatrix}2\\3\end{bmatrix} = \frac{1}{3}T\begin{bmatrix}2\\1\end{bmatrix} + \frac{4}{3}T\begin{bmatrix}1\\2\end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix}5\\1\end{bmatrix} + \frac{4}{3}\begin{bmatrix}4\\5\end{bmatrix}$$

$$= \begin{bmatrix}\frac{5}{3}\\\frac{1}{3}\end{bmatrix} + \begin{bmatrix}\frac{16}{3}\\\frac{20}{3}\end{bmatrix} = \begin{bmatrix}7\\7\end{bmatrix}.$$

Alternative solution (using the answer to part (e)):

$$T\begin{bmatrix}2\\3\end{bmatrix} = T\begin{bmatrix}1\\2\end{bmatrix} + T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix} + \begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}7\\7\end{bmatrix}.$$

3. (a) $\begin{bmatrix}-2 & 1\\-2 & 1\end{bmatrix}$ evidently maps F_1 onto itself. To find the fate of F_2, we can proceed as follows:

Any vector of F_2 is of the form

$$\begin{bmatrix}t\\2t\end{bmatrix},$$

or, equivalently,

$$t\begin{bmatrix}1\\2\end{bmatrix}, \quad \text{where } t \in R.$$

Now,

$$\begin{bmatrix}-2 & 1\\-2 & 1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},$$

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so that the image of F_2 is merely the point $(0, 0)$. Mathematicians sometimes express this by saying that F_2 has been annihilated by the transformation $\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}$.

Next question: What is the fate of F_3? Any vector in F_3 can be written

$$\begin{bmatrix} t \\ -2t \end{bmatrix},$$

or, equivalently,

$$t\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \text{where} \quad t \in R.$$

Now,

$$\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix},$$

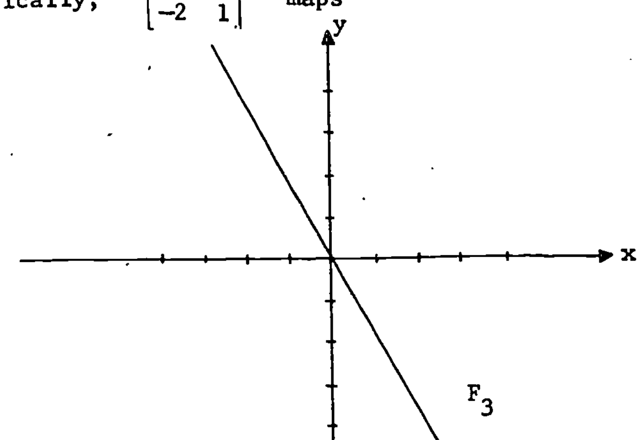
and hence

$$T\begin{bmatrix} t \\ -2t \end{bmatrix} = (4t)\begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Consequently, the image of F_3 is the one-dimensional vector space

$$r\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad r \in R.$$

Geometrically, $\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}$ maps



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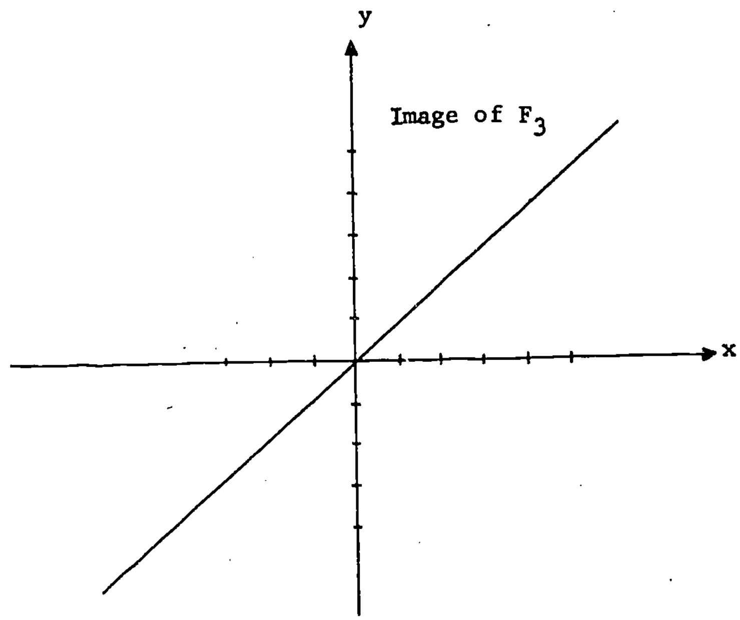
into



Finally, what is the image of H under the transformation $\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}$?

Any vector of H can be written

$$\begin{bmatrix} x \\ y \end{bmatrix},$$

$x \in R$, $y \in R$, and, consequently

$$\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2x + y \\ -2x + y \end{bmatrix} = (-2x + y) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Geometrically, the entire two-dimensional space H is mapped onto the one-dimensional space

$$r \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad r \in R,$$

as shown at the top of page 196 of this Commentary.

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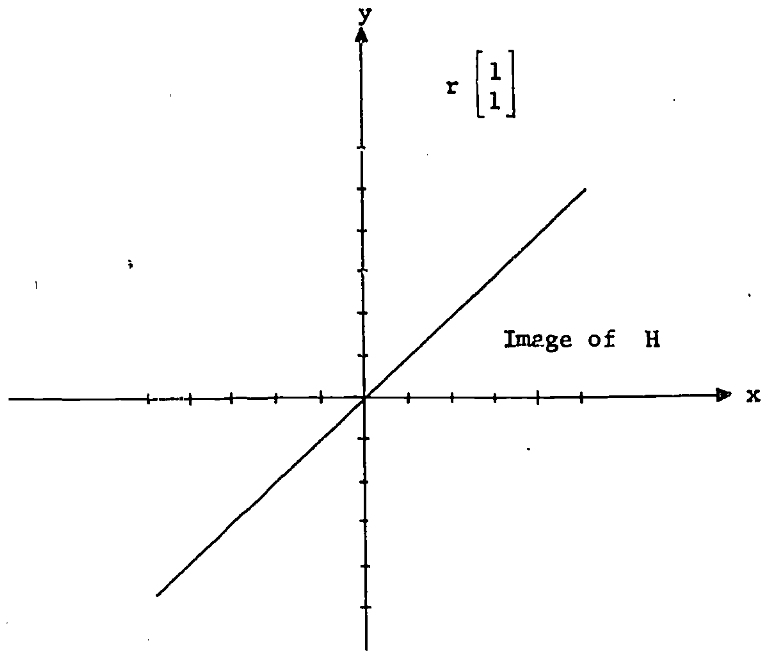


Remark: Any students seriously interested in this topic might find it worthwhile to use the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as basis bectors for H. The result is very suggestive. Can you see a general method that emerges from this train of thought?

(b) Clearly, the matrix B maps F_1 into itself.

What about F_2? Again, any vector of F_2 is of the form

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad t \in R.$$

Consequently,

$$\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

and F_2 is mapped into itself! (We say that F_2 is an invariant subspace under the transformation induced by matrix B.)

For F_3, we have vectors of the form

$$t\begin{bmatrix}1\\-2\end{bmatrix}, \qquad t \in R,$$

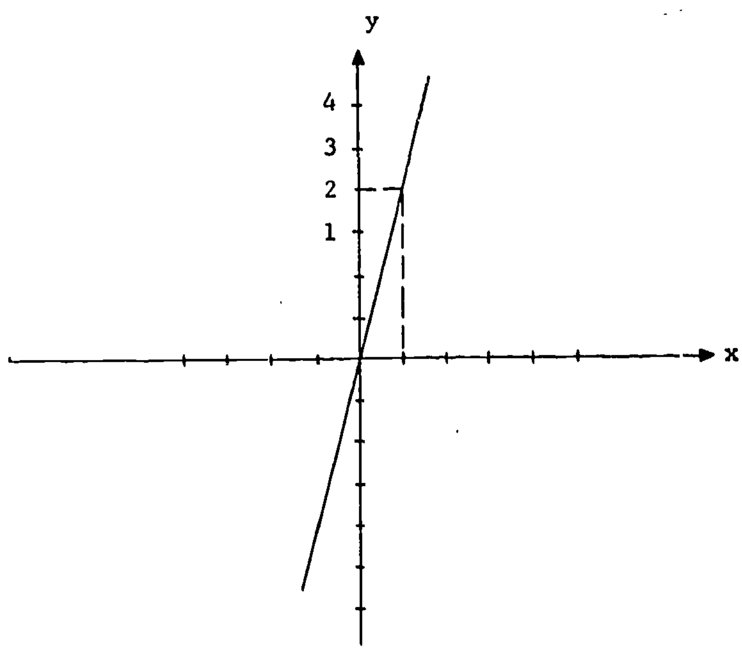
and we see that

$$\begin{bmatrix}0 & 1\\-2 & 3\end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}-2\\-8\end{bmatrix} = (-2)\begin{bmatrix}1\\4\end{bmatrix}.$$

Hence, the image of F_3 is

$$r\begin{bmatrix}1\\4\end{bmatrix}, \qquad r \in R,$$

which forms a 1-dimensional vector space, namely



What does matrix B do to H itself? Since $\delta(B) \neq 0$, B maps H onto itself. (We leave it to you to decide how.)

(c) and (d): Using the matrix AB is equivalent to mapping the space <u>first</u> with B alone, and then mapping the <u>result</u> with A alone.

Actually, it is easier to answer question (d) first:

(d) $\left[\text{Done first for convenience; part (c) follows}\right]$:

Using the matrix BA is equivalent to mapping the space <u>first</u> using A alone, and then taking this result and mapping it using B alone. This is a consequence of the associative law for matrix multiplication:

$$(BA)V = B(AV).$$

The left-hand side, of course, corresponds to using the matrix BA.

The right-hand side says that we take the vector V and transform it by using matrix A. The result of this will be AV. We now take this result, and transform it using matrix B:

$$AV \longrightarrow BAV.$$

The subspace F_1, of course, is mapped onto itself.

How about F_2? We know that A maps F_2 onto (0, 0), i.e., onto F_1. (We have now found (AV).) What happens when we <u>now</u> apply B? Answer: B maps (0, 0) (or F_1) onto itself.

Consequently, BA maps F_2 onto F_1, i.e., into the single point (0, 0).

How about F_3? First, we know that A maps F_3 onto the line $y = x$ (or, if you prefer, onto the 1-dimensional vector space $r\begin{bmatrix}1\\1\end{bmatrix}$, where $r \in R$). Now, what does B do to $\begin{bmatrix}1\\1\end{bmatrix}$?

$$\begin{bmatrix}0 & 1\\-2 & 3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix},$$

so B leaves $\begin{bmatrix}1\\1\end{bmatrix}$, and hence every vector in the space $r\begin{bmatrix}1\\1\end{bmatrix}$, unchanged. Thus, BA maps F_3 in precisely the same way that A alone did.

Finally, what happens to F_4 (i.e., to H itself)? Let us follow our earlier suggestion, and express every vector of H as a linear combination of the basis vectors

$$\begin{bmatrix}1\\2\end{bmatrix} \text{ and } \begin{bmatrix}1\\1\end{bmatrix}.$$

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(We clearly can do this, since these two vectors are not collinear.) Now, if V is any vector in H, then

$$V = a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}1\\1\end{bmatrix}.$$

Applying the matrix A to V, we have

$$\begin{aligned} AV &= A\left(a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}1\\1\end{bmatrix}\right) \\ &= aA\begin{bmatrix}1\\2\end{bmatrix} + bA\begin{bmatrix}1\\1\end{bmatrix} \\ &= \underline{0} + b\begin{bmatrix}-1\\-1\end{bmatrix}, \end{aligned}$$

so that H maps onto the line $y = x$. This line, however, is unaffected by the application of matrix B. Consequently, BA maps H in precisely the same way that A alone did.

(c) [For convenience, the solution to part (d) <u>precedes</u> this solution of part (c).]

Evidently, AB maps F_1 into itself.

What about F_2? We know that B maps F_2 into itself (each vector being multiplied by 2). Then, A maps this result [as we saw in part (a)] into the single point $(0, 0)$. Consequently, AB maps F_2 into $(0, 0)$.

How about F_3 ? B maps F_3 into the vector space

$$r\begin{bmatrix}1\\4\end{bmatrix}, \qquad r \in R,$$

and then A maps $r\begin{bmatrix}1\\4\end{bmatrix}$ as follows:

$$\begin{bmatrix}-2 & 1\\-2 & 1\end{bmatrix}\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix} = 2\begin{bmatrix}1\\1\end{bmatrix}.$$

Consequently, the image of F_3 under the combined transformation induced by AB is the one-dimensional vector space

$$r \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad r \in R$$

(i.e., the line $y = s$).

Finally, how about H itself? We know that B maps H onto itself, and A then maps H onto the line $y = x$. Consequently, the combined transformation maps H onto the line $y = x$.

4. (a) [illegible] $= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p + q \\ q \end{bmatrix}.$$

(b) Here we are given the image vector Z, such that

$$AV = Z.$$

Evidently, this situation calls for A^{-1}:

$$A^{-1} AV = A^{-1} Z,$$

$$IV = A^{-1} Z,$$

$$V = A^{-1} Z.$$

So, we compute A^{-1}:

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Now we are ready for business:

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} r - s \\ s \end{bmatrix}.$$

5. (a) Not linear. We give a counter-example:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

But every linear transformation takes $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ into itself!

(b) Not linear. We give a counter-example:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ should go (by linearity) into $2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

But does it?

Answer

$$2\begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

So the transformation is not linear.

(c) Linear.

(d) Linear.

(e) Not linear.

(f) Not linear. We give a counter-example:

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} \rightarrow \begin{bmatrix} 15 \\ 20 \end{bmatrix}.$$

$2\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ should, by linearity, transform into $\begin{bmatrix} 30 \\ 40 \end{bmatrix}$.

But does it?

$$\begin{bmatrix} 6 \\ 8 \end{bmatrix} \rightarrow \begin{bmatrix} 60 \\ 80 \end{bmatrix}.$$

So the transformation is not linear.

6. If A maps the entire plane into $(0, 0)$, then it must map

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

into $(0, 0)$. Consequently, if we let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we know that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence $a = 0$ and $c = 0$.

To show that b and d must be 0, consider in a similar way the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

7. If A maps <u>every</u> vector into itself, then it must map

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

onto itself.

Consequently, we know that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so that $a = 1$, $c = 0$.

To determine b and d, consider in a similar way the vector

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

8. The line $y = 0$ can be represented as the 1-dimensional vector space

$$r\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad r \in R.$$

Now,

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Hence the line $y = 0$ is mapped onto itself. Every vector is doubled; hence,

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is the only vector mapped onto itself.

9. (a) Follows from direct calculation.

(b) Let A be the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We know that A must map $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ onto the x axis:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so that $c = 0$.

Similarly, A maps $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ onto the x axis:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so that $d = 0$.

Hence A must be of the form

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}.$$

Does this perform as desired? Yes it does, since

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ 0 \end{bmatrix},$$

so A maps H onto the x axis.

10. Proceed similarly (compare Exercise 9). The result is

$$\begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}.$$

11. (a) If $AV = 2V$ for all V, then this must hold for each particular instance, and so we look for some instances that will yield simple computations.

One such "special case" is this: Let V be

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Consequently, $a = 2$, and $c = 0$.

Similarly, by considering

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

we find that $b = 0$, $d = 2$.

(b) Proceed as in part (a). The result is

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$

12. Note that we cannot use A^{-1} in this proof since it is entirely possible that $\delta(A) = 0$.

Instead, we proceed as follows: Let

$$U, V \longrightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then

$$r\,U \longrightarrow r\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Further,

$$U + V \longrightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This completes the proof.

13. Here is the proof:

First, if f is linear, then

$$\begin{aligned} f(rV + sU) &= f(rV) + f(sU) \\ &= rf(V) + sf(U), \end{aligned}$$

so the equation is a <u>necessary</u> condition for linearity.

Conversely, if f satisfies the equation

$$f(rV + sU) = rf(V) + sf(U),$$

then, in particular,

$$f(A + B) = f(A) + f(B)$$

(by substituting $r = s = 1$, $A = V$, $B = U$); also,

$$f(tV) = tf(V)$$

(by substituting $s = 0$, etc.).

This completes the proof,

(In other words, this single equation is equivalent to the two conditions stated in Definition 5-2 on page 190.)

5-3. Linear Transformations

The content of Section 5-3 is very important and useful. We note here just one aspect of it, namely: It provides us with two valuable tools for finding matrices from transformations and vice versa. In both cases, we are able to replace an apparently "hard" problem by a much simpler one.

First, we can study AV for all V by studying the transforms of only two vectors — namely, any two noncollinear vectors. (This method is used to study the rotation transformation on page 198.)

Second, we can sometimes simplify a problem by regarding a transformation as the result of a sequence of simpler transformations. (This method is used to construct the matrix on page 199, bottom.)

We have also used both of these methods in solving some of the problems in the preceding sections.

Exercises 5-3

1. (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$

(b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

(c) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ ay \end{bmatrix}.$

(d) $\begin{bmatrix} -a & 0 \\ 0 & -a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -ax \\ -ay \end{bmatrix}$.

(e) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$.

(f) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ y \end{bmatrix}$.

(g) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$.

(h) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ -y \end{bmatrix}$.

(i) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} 2x \\ y \end{bmatrix}$.

(j) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} 3x \\ 3y \end{bmatrix}$.

(k) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x + y \\ y \end{bmatrix}$.

(l) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ 2x + y \end{bmatrix}$.

(m) $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x - 2y \\ y \end{bmatrix}$.

(n) $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y - 3x \end{bmatrix}$.

2. (a) p is represented by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(b) q is represented by $\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$.

(c) We know that r carries $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ into $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Consequently,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

or

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

so $a = 0$, $c = 1$.

Similarly,

$$\begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

or

$$\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

so $b = -1$, $d = 0$, and the matrix is:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(d) Evidently, s takes

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

into

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so that $a = 1$, $c = 1$.

Similarly, s takes

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

into itself, so that $b = 0$, $d = 1$, and the matrix must be

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

(e) We know that pq corresponds to "first q, then p". Now, q maps H by horizontal projection into $y = -x$, after which p reflects this line in the x axis. The combined matrix can be found as follows:

$$\text{matrix for } q \text{ alone:} \quad \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

Modifying this to include a subsequent reflection in the x axis, we get the matrix

$$\begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}.$$

(We can, of course, also obtain this from the matrix product

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}.)$$

(f) $\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}.$

(g) r takes $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ into $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, after which p carries $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ into $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$. The combined effect, then, is to carry $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ into $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Similarly, r takes $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ into $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$, after which p carries $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ into itself. The combined effect, therefore, is to carry $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ into $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

We know that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$, so right-multiplication by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ merely reproduces the first column $\begin{bmatrix} a \\ c \end{bmatrix}$. Similarly, right-multiplying by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ merely reproduces the second column, $\begin{bmatrix} b \\ d \end{bmatrix}$.

Hence, the matrix for pr must be:

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

<u>Alternative solution</u>: The solution for part (g) that we have just given is one of these "clever" solutions that are somewhat involved to explain, but can actually be done with extremely little computation.

The following solution is "more straightforward" (if you like it), or "more tedious and less exciting" (if you prefer to disparage it):

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

(h) First, p carries $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ into itself, after which r carries $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ into $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence the combined effect is to carry $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ into $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Similarly, p carries $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ into $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$, after which r carries $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ into $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The combined effect, then, is to carry $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ into $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Consequently (cf. the solution to part (g)), the matrix must be

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

<u>Alternative solution</u>: As in part (g), you can find the result by multiplying the matrices for r and p:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(i) $\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}.$

(j) If you think carefully about the geometric effect of sq, you can get this matrix by inspection:

$$\text{for } q \text{ alone:} \quad \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

After q is applied, every point is on the line $y = -x$. We now shift each point vertically through a displacement equal to the x coordinate of the point, i.e., equal to the "opposite" (cf. S.M.S.C., First Course in Algebra) of the y coordinate. This makes no change in the x coordinate; but a number plus its "opposite" (i.e., additive inverse) is zero, so the new y coordinate must be zero, and the matrix is

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

(k) Let's do this the "dull, routine" way:

$$\underset{s}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}} \left(\underset{r}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}} \; \underset{s}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}} \right) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}.$$

(l) By the associative law for matrix multiplication, this must be the same as part (k).

(m) We know that the matrix for sq is

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

If, now, we follow this by a reflection in the x axis, the x coordinate is left unchanged, and the final matrix must be

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

(Another way to say it is: sq maps the entire 2-dimensional space H onto the x axis. But following sq by p can have no further effect, since the points on the x axis are left unchanged by a reflection in that axis; consequently, the combined effect of sq followed by p must be the same as the effect of sq alone.

<u>Alternative solution</u>: One can, of course, multiply the matrices for p and for sq:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

(n) By the associative law for matrix multiplication, this must be the same as part (m).

(o) The geometric interpretation here can help us avoid considerable computation.

First we apply q: This maps H onto the line $y = -x$, by the matrix

$$\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

We follow this by r, a counterclockwise rotation of 90°, so that the points at

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

move (under q) to,

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

respectively, and then (under r) to

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

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We next apply p, so these points move on to

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Finally, we apply s, getting

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Hence, the combination (sp) (rq) takes

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

into

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

respectively. Using our rule about the reproduction of columns when we right-multiply by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we can immediately write the matrix for (sp) (rq) as

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

3. Matrix representing 15° counter-clockwise rotation: either fg or gf. Note: In this case the matrices must commute, as we see at once from their geometric meanings without the need to do any computation whatsoever!

4. (a) If T is a linear transformation, then

$$\begin{aligned} T\vec{0} &= T(\vec{V} - \vec{V}) \\ &= T\vec{V} - T\vec{V} \\ &= \vec{0}. \end{aligned}$$

(b) Let S be a subspace of H, and let T be a linear transformation. Let S' be the image of S under the transformation T. We must show that S' is closed under vector addition and under multiplication by a scalar.

First, closure under vector addition: Let V' and W' belong to S'. Then, by the definition of S', there exist two vectors V and W in S, such that

$$TV = V' ,$$
$$TW = W' .$$

Now, since S is a subspace, we know that $V + W$ belongs to S. Therefore, by the definition of S', $T(V + W)$ belongs to S'.

However,

$$T(V + W) = TV + TW = V' + W' ,$$

which shows that the sum $V' + W'$ belongs to S', and so S' is closed under vector addition.

The proof of closure under multiplication by a scalar is similar.

5. We must prove that

$$(f + g)(aV + bW) = a(f + g)(V) + b(f + g)(W).$$

By the definition of $(f + g)$,

$$\begin{aligned}(f + g)(aV + bW) &= f(aV + bW) + g(aV + bW)\\ &= af(V) + bf(W) + ag(V) + bg(W),\end{aligned}$$

the second equality holding because of the linearity of f and g.

Continuing, we have

$$\begin{aligned}(f + g)(aV + bW) &= af(V) + bf(W) + ag(V) + bg(W)\\ &= af(V) + ag(V) + bf(W) + bg(W)\\ &= a\left[f(V) + g(V)\right] + b\left[f(W) + g(W)\right]\\ &= a(f + g)(V) + b(f + g)(W).\end{aligned}$$

The reasons for these last three steps are, respectively, the commutative and associative laws for real numbers, the distributive law for real numbers, and the definition of $(f + g)$.

6. The proof is similar to that in Exercise 5, above.

7. We prove that fg is represented by AB. This implies that fg is linear.

Proof that fg is represented by AB:

Let A and B be, respectively,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} e & h \\ k & m \end{bmatrix}.$$

By our rule for the reproduction of columns when we right-multiply by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (cf. solution to Exercise 2 in this set), we know that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

are carried into

$$\begin{bmatrix} e \\ k \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} h \\ m \end{bmatrix},$$

respectively, by matrix B (i.e., by transformation g).

If we now apply transformation f (i.e., matrix A), the vectors

$$\begin{bmatrix} e \\ k \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} h \\ m \end{bmatrix}$$

will be carried into

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e \\ k \end{bmatrix} = \begin{bmatrix} ae + bk \\ ce \quad dk \end{bmatrix},$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} h \\ m \end{bmatrix} = \begin{bmatrix} ah + bm \\ ch \quad dm \end{bmatrix},$$

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respectively.

Combining the two transformations, we see that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are carried by fg into

$$\begin{bmatrix} ae + bk \\ ce + dk \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} ah + bm \\ ch + dm \end{bmatrix},$$

respectively.

Using our rule about the reproduction of columns when we right-multiply by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we can write a matrix that will take $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ into

$$\begin{bmatrix} ae + bk \\ ce + dk \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} ah + bm \\ ch + dm \end{bmatrix},$$

respectively.

This matrix must be

$$C = \begin{bmatrix} ae + bk \\ ce + dk \end{bmatrix} \begin{bmatrix} ah + bm \\ ch + dm \end{bmatrix}.$$

By multiplying AB, we find that

$$AB = C.$$

Now, the matrix C maps $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ correctly. However, we cannot infer from this that C maps every vector correctly (i.e., the same way that AB does), since we do not yet know that fg is linear.

To complete the proof, observe that fg is linear if and only if the matrix C (which we know must induce a linear transformation, since every matrix does; cf. Theorem 5-5, p. 209) maps <u>every</u> vector the same way that fg does, i.e., for all vectors V,

$$A(BV) = CV.$$

But this must be true, in consequence of the fact that $(AB) = C$, and that matrix multiplication is associative:

$$A(BV) = (AB)V = CV.$$

This completes the proof for fg. The proof for gf is similar, or can be obtained from this proof by relabeling.

8\. (a) We must prove that, for all vectors V belonging to H, we have

$$f(g + h)V = (fg + fh)V. \tag{1}$$

To establish the validity of equation (1), we use the definitions of sum and product of transformations as follows.

By definition,

$$\begin{aligned}(fg + fh)V &= (fg)V + (fh)V \\ &= f(gV) + f(hV).\end{aligned} \tag{2}$$

Also, by definition,

$$f(g + h)V = f\left[gV + hV\right].$$

Because f is a linear transformation, we have

$$f\left[gV + hV\right] = f(gV) + f(hV).$$

Comparison with equation (2) now establishes the validity of equation (1). This completes the proof.

(b) We must prove that, for all vectors V belonging to H,

$$\left[(f + g)h\right] V = (fh + gh)V.$$

From the definition of the sum and product of transformations, we have

$$(fh + gh)V = (fh)V + (gh)V = f(hV) + g(hV) = (f + g)(hV) = \left[(f + g)\right] hV.$$

This completes the proof.

(c) In each part of Exercise 8, the important thing is to remember that we are working with linear transformations, not with matrices. The formulae look similar, but have different meanings. Thus, for example, if A, B, and V are matrices, then

$$A(BV) = (AB)V$$

is an instance of the associative law for matrix multiplication.

On the other hand, if f and g are linear transformations, the statement

$$f(gV) = (fg)V$$

is the definition of the product of two linear transformations.

To prove part (c), we must show that, for all vectors V of H,

$$f\left[(ag)V\right] = a(fg)V.$$

But this is easy:

$$a(fg)V = (af)(gV) = a\left[f(gV)\right] = f\left[a(gV)\right] = f\left[(ag)V\right].$$

7-4. One-to-one Linear Transformations

The text makes little use of the concept of dimension of a vector space. Because this is a rather sophisticated (and deliberately open-ended) course, we have not hesitated to introduce this concept, even though briefly.

In particular, we use it to solve Exercises 6 and 7, though alternative proofs can be given.

"Dimension" is an intuitively simple idea that often lends itself to easily visualized proofs.

Here are some fundamental theorems and definitions:

Theorem 1 on Dimension. Let K be a vector space; let $U_1, U_2, \ldots, U_n$ be a linearly independent set of vectors with the property that every vector in K can be expressed in the form

$$a_1 U_1 + a_2 U_2 + \cdots + a_n U_n,$$

for some set of real numbers $a_1, a_2, \ldots, a_n$; and let $V_1, V_2, \ldots, V_m$ be another linearly independent set of vectors, again with the property that every vector in K can be expressed as a linear combination

$$b_1 V_1 + b_2 V_2 + \cdots + b_m V_m,$$

for real numbers $b_1, b_2, \ldots, b_m$. Then $m = n$.

<u>Proof</u>. One can prove this rather easily by supposing that $m < n$, expressing each of the U_i in terms of the V_i, and showing that this contradicts the linear independence of the U_i. Therefore we have $m \geq n$, and similarly $n \geq m$. Therefore, $n = m$.

(To make matters easy, if you present this proof in class, you might try the case $n = 3$, $m = 2$, which you can easily write out explicitly in complete detail.)

<u>Definition of Dimension</u>. The number n of Theorem 1 above will be called the dimension of K.

<u>Theorem 2 on Dimension</u>. Let T be a linear transformation that maps the vector space K into the vector space L. Then the dimension of the range is less than or equal to the dimension of the domain.

<u>Proof</u>. The proof is straightforward, by assuming the contrary and using the properties of T that

$$T(a_1 U_1 + a_2 U_2 + \cdots + a_n U_n)$$
$$= a_1 T U_1 + a_2 T U_2 + \cdots + a_n T U_n$$

and $T\,\underline{0} = \underline{0}$.

<u>Exercises 5-4</u>

1. (a) Is one-to-one.

   (b) Is not one-to-one.

   (c) Is one-to-one.

(d) Is one-to-one.

(e) Is not one-to-one.

(f) Is not one-to-one.

(g) Is one-to-one.

(h) Is one-to-one.

(i) Is one-to-one.

(j) Is one-to-one.

(k) Is one-to-one.

(l) Is one-to-one.

(m) Is one-to-one.

(n) Is one-to-one.

2. Suppose T is <u>not</u> one-to-one. Then there exist two vectors V and W, not equal, such that

$$T\,V = T\,W. \tag{1}$$

Denote $T\,V$ by the letter Z. We can rewrite (1) as

$$T\,V = Z,$$

$$T\,W = Z.$$

Now, $T(V - W) = Z - Z = \underline{0}$; but $V - W$ is not $\underline{0}$. Hence, if T is <u>not</u> one-to-one, the kernel does <u>not</u> consist solely of the zero vector.

These statements are somewhat complicated, and this may be a good place to write out our logic symbolically:

Let I be the statement:

T is one-to-one.

Then $\sim I$ is the statement:

T is not one-to-one.

Let K be the statement:

The kernel consists only of the zero vector. Then $\sim K$ is the statement:

The kernel does not consist solely of the zero vector.

There are four possible forms, representing actually two statements: first,

$$I \Rightarrow K \quad \text{and its equivalent} \quad \sim K \Rightarrow \sim I;$$

second,

$$K \Rightarrow I \quad \text{and its equivalent} \quad \sim I \Rightarrow \sim K.$$

We have already established the second of these two statements (in its equivalent form $\sim I \Rightarrow \sim K$).

We must now prove the first statement.

Suppose that the kernel does not consist of the zero vector alone. Then there exists a vector R such that

$$T\,R = \underline{0},$$

but R itself is not the zero vector.

Then T cannot be one-to-one because, for any vector V, we have

$$T(V + R) = T\,V + T\,R = T\,V + \underline{0} = T\,V,$$

and the distinct vectors V and V + R have identical images. This establishes the implication

$$\sim K \Rightarrow \sim I,$$

or its equivalent,

$$I \Rightarrow K.$$

This completes the proof of the theorem.

(The use of symbolic logic is a matter of taste. In our experience, it does simplify exposition once students become accustomed to it.)

3. This proof follows easily by direct computation.

4. We must prove that

$$fg\ V = fg\ W$$

implies

$$V = W.$$

Since f is one-to-one, we know that

$$f(g\ V) = f(g\ W)$$

implies

$$g\ V = g\ W,$$

and the required result now follows from the fact that g is one-to-one.

5. The result that the set of transformations is a group follows easily from Exercise 4, plus the fact that matrix multiplication is associative.

6. Since no linear transformation can increase dimension (i.e., the dimension of the range is less than or equal to the dimension of the domain), if either f or g decreases dimension (i.e., is not one-to-one), then fg must decrease dimension (and hence fail to be one-to-one).

7. If A is the matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

it evidently maps onto a point.

If A is the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then we know that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}.$$

Hence, if the two columns of A constitute linearly independent vectors, then A maps onto H. In this case, we know that $\delta(A) \neq 0$.

Hence, if $\delta(A) = 0$, the columns of A are dependent; i.e., A is of the form

$$\begin{bmatrix} a & ka \\ c & kc \end{bmatrix}.$$

But A of this form implies that

$$\begin{bmatrix} a & ka \\ c & kc \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a(x + ky) \\ c(x + ky) \end{bmatrix}$$

$$= m\begin{bmatrix} a \\ c \end{bmatrix},$$

and therefore, unless $a = c = 0$, the range is a line.

(b) Follows by direct expansion of the matrix product.

(c) Since A is not the zero matrix, and $\delta(A) = 0$, we know that the range is a one-dimensional vector space, i.e., a line. The kernel must therefore itself be a one-dimensional vector space. This proves the required result.

(d) This is an important result that clarifies the situation of two simultaneous equations in two unknowns when the determinant of the coefficient matrix is zero.

If U does not belong to the range of A (which we know to be one-dimensional), then evidently

$$AV = U$$

can have no solutions.

If, however, U does belong to the range of A, then we must show that

$$AV = U$$

if and only if $V \in \{V_1 + tV_2 \mid t \in R, AV_1 = U, AV_2 = 0\}$.

First, if V belongs to this set, then, by the linearity of A, it must be true that

$$AV = A(V_1 + tV_2) = AV_1 + tAV_2 = U + \underline{0} = U.$$

Suppose, on the other hand, that

$$AV = U.$$

Consider the vector $V - V_1$. Then we have

$$A(V - V_1) = AV - AV_1 = U - U = \underline{0},$$

and hence $V - V_1$ belongs to the kernel of A. Since, however, the kernel of A is one-dimensional, we have $V - V_1 = mV_2$ for some real number m, and this completes the proof.

8. If A^{-1} exists, then

$$AV = U$$

implies

$$V = A^{-1} U,$$

and hence V is uniquely determined.

5-5. Characteristic Values and Characteristic Vectors

The notion of a fixed element for a transformation is extraordinarily useful in mathematics, yet it is deceptively simple.

You might begin by asking if somewhere on each line in the coordinate plane there is a point that has the same value for its ordinate as it has for its abscissa. On the line given by

$$y = 2x + 3,$$

such a point is $(-3, -3)$. On the horizontal line

$$y = 5,$$

there is $(5,5)$; and $(1,1)$ is on the vertical line

$$x = 1.$$

The first of the foregoing examples can be looked at as follows: In the transformation

$$x \longrightarrow 2x + 3,$$

the value $x = -3$ is carried to the same value -3. That is why -3 is called a fixed value for the transformation.

The class will soon discover that the problem of determining the fixed values for the transformation

$$x \longrightarrow ax + b$$

is simply the problem of solving the equation

$$x = ax + b,$$

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and will see that this is always possible if $a \neq 1$. The situation for $a = 1$ becomes clear from a consideration of such equations as

$$x = x$$

and

$$x = x + 5.$$

Perhaps they will want to look for fixed values in the transformation

$$x \longrightarrow ax^2 + bx + c.$$

Exercises 5-5

1. (a) $\begin{vmatrix} 2 - c & 5 \\ 0 & 3 - c \end{vmatrix} = 0,$

$$(2 - c)(3 - c) = 0,$$
$$c = 2,3.$$

For the characteristic value $c = 2$, we have

$$\begin{bmatrix} 0 & 5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

One characteristic vector is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and all others are of the form

$$m\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For the characteristic root $c = 3$, we get

$$\begin{bmatrix} -1 & 5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which is equivalent to

$$-a + 5b = 0,$$

One characteristic vector is

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix},$$

and all others are of the form

$$n\begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

(b) $\begin{vmatrix} -3 - c & 4 \\ -1 & 2 - c \end{vmatrix} = 0,$

$$(-3 - c)(2 - c) + 4 = 0,$$

$$c^2 + c - 2 = 0,$$

$$(c + 2)(c - 1) = 0,$$

$$c = 1, -2.$$

For the characteristic root $c = 1$, we have

$$\begin{bmatrix} -4 & 4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which is equivalent to

$$-4x + 4y = 0,$$
$$-x + y = 0.$$

One characteristic vector is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and all others are of the form

$$\begin{bmatrix} k \\ k \end{bmatrix}, \qquad k \in R,\ k \neq 0.$$

For the characteristic root $c = -2$, we find

$$\begin{bmatrix} -1 & 4 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which is equivalent to

$$-x + 4y = 0,$$
$$-x + 4y = 0.$$

One characteristic vector is

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix},$$

and all others are of the form

$$\begin{bmatrix} 4k \\ k \end{bmatrix}, \qquad k \in R,\ k \neq 0.$$

(c) $\begin{vmatrix} 2 - c & 1 \\ -1 & -c \end{vmatrix} = 0,$

$$-2c + c^2 + 1 = 0,$$

$$(c - 1)^2 = 0,$$

$$c = 1.$$

Here, the characteristic equation has a "double" root. What kind of new behavior will this entail?

We look for a characteristic vector:

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$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The vector

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

is a characteristic vector, and all other characteristic vectors are scalar multiples of this one.

Can we not find any linearly independent characteristic vectors? The answer in this case is "No," but if we look for vectors that, instead of being carried into

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

by the matrix

$$M = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},$$

are carried into zero by M^2, we <u>are</u> able to find a set of basis vectors. Thus

$$\begin{aligned} M^2 &= \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

which carries <u>all</u> vectors into zero.

A set of basis vectors can now be chosen in a very special way:

We construct a "chain of length 2" by selecting any convenient vector that is carried into

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

by H^2; for example, we might choose

$$V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We now pick, for our second vector, the image of V_1 under M:

$$M\,V_1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad V_2.$$

If we use the vectors

$$M\,V_1, \quad V_1$$

as basis vectors, in that order, the matrix M will assume "triangular form":

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

For a complete discussion of this important phenomenon, see Bernard Friedman, Principles and Techniques of Applied Mathematics, (Wiley 1956).

(d) $\begin{vmatrix} -c & 2 \\ 0 & 1-c \end{vmatrix} = 0,$

$$c^2 - c = 0,$$

$$c = 0,\ 1.$$

For $c = 0$, we have

$$\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The characteristic vectors are all scalar multiples of

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For $c = 1$, we get

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

i.e.,

$$-x + 2y = 0.$$

One characteristic vector is

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

and all others are scalar multiples of this one.

2. This follows almost immediately. Here is the proof:

If zero is a characteristic value, then

$$\begin{vmatrix} a-0 & b \\ c & d-0 \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0,$$

since otherwise there could be no nonzero vector

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Reversing this line of argument, we obtain the converse result.

3. First, f is one-to-one if and only if its kernel (i.e., the set of vectors mapped into the zero vector) consists only of the zero vector.

But the equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has a nontrivial solution if and only if the determinant $\delta(A) = 0$, i.e., if and only if 0 is a characteristic root.

4. $$\begin{vmatrix} 6 - c & 2 \\ 2 & 3 - c \end{vmatrix} = 0,$$

$$c^2 - 9c + 14 = 0,$$

$$(c - 7)(c - 2) = 0,$$

$$c = 7,\ 2.$$

For $c = 7$, we get

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

One characteristic vector is

$$V = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

and the corresponding fixed line is

$$mV, \quad m \in R.$$

For $c = 2$, we have

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

with the characteristic vector

$$W = \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

and the fixed line

$$mW, \quad m \in R.$$

Finally, the inner product

$$V \bullet W = (1)\ (2) + (-2)\ (1) = 0$$

shows that V and W are perpendicular.

5. $$A^2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{11}a_{21} + a_{22}a_{21} & a_{21}a_{12} + a_{22}^2 \end{bmatrix},$$

$$A^2 - (a_{11} + a_{22})A + \delta(A)I$$

$$= \begin{bmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{11}a_{21} + a_{22}a_{21} & a_{21}a_{12} + a_{22}^2 \end{bmatrix} - \begin{bmatrix} a_{11}^2 + a_{22}a_{11} & a_{11}a_{12} + a_{12}a_{22} \\ a_{11}a_{21} + a_{22}a_{21} & a_{11}a_{22} + a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}a_{22} - a_{12}a_{21} & 0 \\ 0 & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

as desired.

6. The first part follows by direct calculation. This does not contradict Theorem 5-11, since the transformation in question is not linear.

7. If A maps every line through the origin into itself, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = r \begin{bmatrix} x \\ y \end{bmatrix}$$

for some $r \in R$, and for <u>all</u> $x,y \in R$.

This is equivalent to

$$\begin{aligned} ax + by &= rx, \\ cx + dy &= ry, \qquad \text{for all } x,y \in R. \end{aligned}$$

Hence $a = r$, $b = 0$, $c = 0$, $d = r$, as desired. The converse is trivial.

8. The characteristic equation for this matrix is

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$$\begin{vmatrix} a_{11} - c & a_{12} \\ a_{21} & a_{22} - c \end{vmatrix} = 0,$$

i.e.,

$$c^2 - (a_{22} + a_{11})c + a_{11}a_{22} - a_{12}a_{21} = 0.$$

This quadratic equation for the "unknown" c will have 2, 1, or 0 real roots according as its discriminant

$$b^2 - 4ac = (a_{22} + a_{11})^2 - 4(a_{11}a_{22} - a_{12}a_{21})$$

is positive, zero, or negative.

However, this can be expanded as

$$a_{22}^2 + 2a_{22}a_{11} + a_{11}^2 - 4a_{11}a_{22} + 4a_{12}a_{21}$$

$$= (a_{22} - a_{11})^2 + 4a_{12}a_{21}$$

$$= (a_{11} - a_{22})^2 + 4a_{12}a_{21} = d.$$

9. $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

10. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

11. $\begin{vmatrix} r - c & s \\ s & t - c \end{vmatrix} = 0,$

$$(r - c)(t - c) - s^2 = 0,$$

$$c^2 - (r + t)c + rt - s^2 = 0.$$

The discriminant of this quadratic equation is

$$\begin{aligned} b^2 - 4ac &= (r + t)^2 - 4rt + 4s^2 \\ &= r^2 + 2rt + t^2 - 4rt + 4s^2 \\ &= (r - t)^2 + 4s^2 > 0 \quad \text{if} \quad s \neq 0. \end{aligned}$$

12. The equation

$$\begin{vmatrix} a - c & b \\ d & e - c \end{vmatrix} = 0$$

can be written

$$(a - c)(e - c) - bd = 0.$$

For the transposed matrix, the characteristic equation is

$$\begin{vmatrix} a - c & d \\ b & e - c \end{vmatrix} = 0,$$

which is precisely the same equation.

5-6. Rotations and Reflections

This concluding section of the chapter ties together and reviews aspects of the present course and of the students' earlier work in Euclidean geometry, analytic geometry, and trigonometry.

Exercises 5-6

1. We know that the required matrix must be of the general form

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

The only question that may lurk in your mind is whether α should be the angle given in each part of Exercise 1, or whether α should be the

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negative of this angle. We can settle this by considering the case where we rotate H through $+90^\circ$. Such a rotation carries

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

into

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

If we try $\alpha = +90^\circ$, we have the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and we find that

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and this is, indeed, the correct choice.

(Had we tried $\alpha = -90^\circ$, we would have obtained

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

as the image of

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

a fact you can easily check for yourself.)

We have thus cleared up any doubt as to which direction of rotation corresponds to which sign for the angle α.

Here are the matrices M:

(a) $\alpha = 180^\circ$, $\cos 180^\circ = -1$, $\sin 180^\circ = 0$, so

$$M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(It is directly evident that this matrix induces a rotation through 180°, since any vector is mapped into its own additive inverse, $V \longrightarrow -V$.)

(b) $\alpha = 45^\circ$, $\cos 45^\circ = \sin 45^\circ = \frac{1}{\sqrt{2}}$, so

$$M = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

(We can clock the image of

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We have

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix},$$

which is evidently correct. How many other vectors should we check to be sure we have a correct matrix? What would be a convenient choice?)

(c) $\alpha = 30^\circ$, $\cos 30^\circ = \frac{\sqrt{3}}{2}$, $\sin 30^\circ = \frac{1}{2}$, so

$$M = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

(d) $\alpha = 60^\circ$, $\cos 60^\circ = \frac{1}{2}$, $\sin 60^\circ = \frac{\sqrt{3}}{2}$, so

$$M = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

We note that a rotation of 30°, followed by a rotation of 60°, should be equivalent to a single rotation through an angle of 90°. We now check this:

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

which does, indeed, check.

(e) $\alpha = 270^\circ$. We can obtain this by rotating through 180°, and then rotating through 90°. Recalling that the matrix on the right corresponds to the first transformation, we can write

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

so this last matrix must correspond to a rotation through 270°.

(Although we have been careful with the left-right order of our matrices, this is unimportant in the present problem. It is evident from the geometrical interpretation that rotation matrices commute. Can you prove this algebraically?)

(f) $\alpha = 90^\circ$. We have already obtained this matrix, namely,

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(g) From the geometric interpretation, it is clear that the matrix

$$R_2(\alpha) \equiv \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{bmatrix}$$

is the matrix inverse of the matrix

$$R_1(\alpha) \equiv \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

Evidently $R_2(\alpha)$ rotates H through an angle of $-\alpha$.

Using the fact that $\cos\theta$ is an <u>even</u> function, whereas $\sin 0$ is an odd function, we get

$$R_2(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}.$$

Consequently, if we find the matrix for $+120^\circ$, we can get the matrix for -120° merely by reversing the signs of the two off-diagonal terms.

But $120^\circ = 90^\circ + 30^\circ$, so we have

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

This is the matrix for $\alpha = +120^\circ$. "Oppositing" the off-diagonal terms, we get the matrix for $\alpha = -120^\circ$, namely,

$$M = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

(h) This, evidently, is the identity transformation, represented by the matrix

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(i) We use $135^\circ = 90^\circ + 45^\circ$, and then opposite the off-diagonal terms (cf. answer to part g):

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} .$$

Oppositing the off-diagonal terms, we get:

$$M = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} .$$

(j) $150^\circ = 90^\circ + 60^\circ$; hence we have

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$$

2. (a) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

The second matrix factor above merely reproduces the left-hand column of the first factor, and opposites the right-hand column of the first factor:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a & -b \\ c & -d \end{bmatrix} .$$

We can now write down the remaining matrices by inspection, using the answers to Exercise 1.

(b) $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

(c) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$.

(d) $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$.

(e) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

(f) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(g) $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

(h) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(i) $\begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

(j) $\begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$.

3. This follows directly by carrying out the matrix multiplication.

4. Let us see, first of all, what we can learn about orthogonal matrices.

Let M be the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} .$$

If the lengths of the vector and its image are equal, we have

$$x^2 + y^2 = a^2x^2 + 2abxy + b^2y^2 + c^2x^2 + 2cdxy + d^2y^2 .$$

For this to hold for all x and all y, we must have

$$\begin{aligned} a^2 + c^2 &= 1, \\ b^2 + d^2 &= 1, \\ ab &= -cd. \end{aligned} \qquad (1)$$

Conditions (1) are necessary and sufficient.

Let us see what happens if we apply this same approach to the transposed matrix,

$$M^t \equiv \begin{bmatrix} a & c \\ b & d \end{bmatrix} .$$

A necessary and sufficient set of conditions for M^t to be orthogonal are:

$$\begin{aligned} a^2 + b^2 &= 1, \\ c^2 + d^2 &= 1, \\ ac &= -bd. \end{aligned} \qquad (2)$$

The \$64,000 question, then, is whether conditions (1) are equivalent to conditions (2).

A very clever way to answer this question is to make use of our knowledge of vectors, trigonometry, and geometry.

We can recast the conditions (1) in a new form by using the vectors

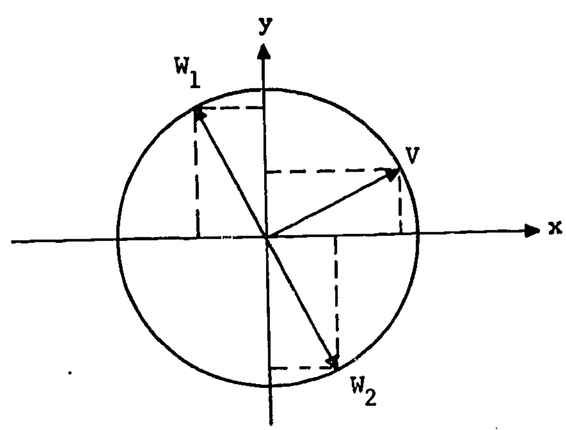
$$V \equiv \begin{bmatrix} a \\ c \end{bmatrix},$$

$$W \equiv \begin{bmatrix} b \\ d \end{bmatrix}.$$

Conditions (1) become

$$|V| = 1,$$
$$|W| = 1, \qquad (1')$$
$$V \bullet W = 0.$$

Hence V and W are orthogonal unit vectors, and they must look somewhat as follows:



(W_1 and W_2 are the possible locations of W, if V is given.)

In order to get from

$$\begin{bmatrix} a \\ c \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b \\ d \end{bmatrix}$$

to the new vectors (appropriate for conditions (2))

$$\begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c \\ d \end{bmatrix},$$

we want to interchange the second component of V with the first component of W. The diagram, however, makes it clear that these components are equal in magnitude, but may perhaps differ in sign. This difference in sign, however, may not interfere in our three equations.

Now that we know approximately where we stand, let us begin with conditions (1) and deduce the equivalence of conditions (2) by a purely algebraic calculations. (Special cases occur if some entries are zero; consideration of these will be left to the reader.)

$$a^2 + c^2 = 1,$$

$$\left(\frac{a}{c}\right)^2 + 1 = \frac{1}{c^2}.$$

But

$$ab = -cd,$$

so we have

$$\frac{a}{c} = -\frac{d}{b},$$

$$\frac{a^2}{c^2} = \frac{d^2}{b^2},$$

and hence

$$\left(\frac{d}{b}\right)^2 + 1 = \frac{1}{c^2}. \qquad (3)$$

However, we have

$$b^2 + d^2 = 1,$$

and so

$$1 + \left(\frac{d}{b}\right)^2 = \frac{1}{b^2}. \qquad (4)$$

Combining (3) and (4) gives us

$$b^2 = c^2,$$

whence

$$b = \pm c.$$

This is the result that we saw geometrically in our heuristic discussion.

Since we have $\frac{a^2}{c^2} = \frac{d^2}{b^2}$, and also $c^2 = b^2$, we can conclude that that $a = \pm d$.

Hence, if V is

$$\begin{bmatrix} a \\ c \end{bmatrix},$$

then W is one of the following:

$$\begin{bmatrix} -c \\ a \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} c \\ -a \end{bmatrix}.$$

(We also saw this result geometrically in our heuristic argument.)

What happens, then, if we consider the pair of vectors

$$\begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c \\ d \end{bmatrix}?$$

These must be

$$\begin{bmatrix} a \\ c \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c \\ -a \end{bmatrix},$$

or else

$$\begin{bmatrix} a \\ -c \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c \\ a \end{bmatrix}.$$

Hence we have seen that, if

$$\begin{bmatrix} a \\ c \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b \\ d \end{bmatrix}$$

are orthogonal unit vectors, then so are

$$\begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c \\ d \end{bmatrix}.$$

The argument is basically symmetric, however, and we can show similarly that, if

$$\begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c \\ d \end{bmatrix}$$

are orthogonal unit vectors, then so are

$$\begin{bmatrix} a \\ c \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b \\ d \end{bmatrix}.$$

Hence the conditions are equivalent.

But

$$\begin{bmatrix} a \\ c \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b \\ d \end{bmatrix}$$

are orthogonal unit vectors if and only if conditions (1) are satisfied, and

$$\begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c \\ d \end{bmatrix}$$

are orthgonal unit vectors if and only if conditions (2) are satisfied.

Hence conditions (1) and conditions (2) are equivalent; that is to say, the matrix M is orthogonal if and only if its transpose M^t is also.

5. In this solution, we shall make use of our discussion (and notation) in the preceding solution.

We know that M is orthogonal if and only if its transpose M^t is also. Suppose that M is orthogonal.

We compute the product $M\,M^t$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix} ;$$

but, using conditions (1'), we get

$$a^2 + b^2 = 1, \quad ac + bd = 0, \quad c^2 + d^2 = 1,$$

and we have

$$M\,M^t = I.$$

Similarly,

$$M^t M = I.$$

Thus, the transpose M^t is the inverse!

Consequently, the result of Exercise 5 follows from the result of Exercise 4.

6. This can be computed easily by observing that any orthogonal matrix (thanks to conditions (1) of Exercise 4) can be written in the form

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

or else in the form

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}.$$

7. (a) This might equally well be given as the definition of "translating H in the direction of the vector U and through a distance equal to the length of U."

(b) First, the vector

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is mapped into the vector

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

which is not of the same length. Then the statement that "the mapping preserves the length of every vector " must be false.

Now, for the other half of this problem: The point (a,b) is mapped into the point $(a + 2, \; b + 3)$, and the point (c,d) is mapped into the point $(c + 2, \; d + 3)$.

The distances between the original two points, and between the image points, are respectively

$$\sqrt{(a - c)^2 + (b - d)^2}$$

and

$$\sqrt{\left[(a + 2) - (c + 2)\right]^2 + \left[(b + 3) - (d + 3)\right]^2} .$$

It is evident that these are equal.

(c) Every linear mapping must carry

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

into itself. This mapping fails to do so, and consequently cannot be linear.

8. This computation follows the same pattern as that in Exercise 6, except that matrices of the form

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$$

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cannot occur.

9. It implies that, if the complex numbers Z_1 and Z_2 are represented as

$$Z_1 = \cos\alpha + i\sin\alpha,$$
$$Z_2 = \cos\beta + i\sin\beta,$$

(note that both Z_1 and Z_2 lie on the unit circle in the complex plane), then

$$W \equiv Z_1 \times Z_2$$

is representable as

$$W = \cos(\alpha + \beta) + i\sin(\alpha + \beta).$$

The correspondence between

$$\begin{pmatrix} \cos\alpha & -\sin\beta \\ \sin\beta & \cos\alpha \end{pmatrix}$$

and

$$\cos\alpha + i\sin\alpha$$

is an isomorphism.

10. (a) A reflection across the line of the vector

$$\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

can be constructed by rotating H through an angle $-\alpha$, whereupon the vector

$$\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

will come to coincide with the vector

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

reflecting across the X axis; and then rotating the space through an angle of $+\alpha$.

The product of the matrices to do this is

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & 2\cos\alpha\sin\alpha \\ 2\cos\alpha\sin\alpha & \sin^2\alpha - \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}.$$

This, then, must be the required matrix.

We can perform an unofficial (but reassuring) check by considering two special cases. If $\alpha = 0$, we have a reflection across the x axis, and the matrix becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which is, in fact, correct.

Suppose $\alpha = \frac{\pi}{2}$. Then we should have a reflection across the y axis, and the matrix should become

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Upon substituting $\alpha = \frac{\pi}{2}$, $\cos \pi = -1$, $\sin \pi = 0$, we find that it does so.

(b) This follows immediately if we substitute $w = \alpha/2$ into Equation (8).

Appendix

RESEARCH EXERCISES

The material in this Appendix is probably too difficult for all but a few extraordinarily talented students. As may be seen from the exercises, all of which are worked out, the bulk of mere manipulation is great. Hence only those students who possess both creative imagination and considerable capacity for pencil pushing are likely to profit from these exercises. Another difficulty with all of this material is that there is no peg on which to hang it — that is to say, the student probably has no background in the arts or sciences to which this mathematics could be applied.

The usual reaction of the mathematical novice to the "hat trick" technique of solution, which is repeatedly used here, is one of puzzled bewilderment. This phrase refers, of course, to the prestidigitator who pulls a rabbit from an apparently empty hat to the astonishment of a naive audience. Now professional mathematicians do not think it bad practice to use the hat trick; in fact, both they and their audiences enjoy this when it is properly performed. It is simply good pedagogy, however, to inform your audience that you are going to do a hat trick. Moreover, the good teacher will have no difficulty in properly preparing his students and arranging for a climax in appreciation and interest at the appropriate time in the discussion. You will perhaps recall Colley Cibber's advice to young actors — first tell your audience what you are going to do, then tell them what you are doing while you are doing it, and finally tell them when you have done it.

Research in any field is best presented as a journey into the unknown. As such, it is fraught with danger, difficulty, and all sorts of pleasant and unpleasant surprises. But also it is quite comparable to the activity undertaken by the creative writer, the artist, and the composer. It may be well to point out this relationship, which is not always obvious to young people.

Let us look at Section 1, dealing with quaternions. It may already have occurred to the imaginative student that there is no real reason for requiring the entries of a matrix to be real numbers. For quaternions, the entries are allowed to be complex numbers. If the students do not notice that restricting these entries to be real numbers will reduce the quaternion to a complex number (Text, pg.94), then this ought to be pointed out to them. It might be well at this time to point out that matrices with complex entries are matrices with entries that are themselves matrices, and that if one liked to do this, one could also consider matrices of matrices of matrices, and so on _ad infinitum_ — or _ad nauseam_.

Just why one should single out matrices of this peculiar quaternion form as an object of intensive study, however, is by no means immediately apparent. If you like, their invention was a stroke of genius on the part of the mathematician W. R. Hamilton. And so indeed was Mozart's _Eine kleine Nachtmusik_ and Shelley's _Ode to a Skylark_.

The material in this section involves straightforward, but lengthy, computation. On pp. 222, 223 of the Text is a mere hint, but the best that could be offered of the way in which the algebra of quaternions is associated with geometry.

The introduction to Section 2 is cursory. It may be mentioned that "Lie" is pronounced "Lee," though perhaps the students may recognize this fact since Lie was the name of the first secretary of the United Nations.

In Section 3, some aspects of the general theory of subsets of 2×2 matrices are developed. Some particular subsets were studied in Chapter 2. Perhaps the present material will encourage students to investigate further the basic mathematics of sets and their subsets.

The last section is much closer to present-day mathematics than the three preceding sections. It has more, perhaps, in the way of elegance and actuality. The introductory paragraph is essentially a factual description of what mathematicians do. It should be stated that the good mathematician delights in constructing an ingenious or original technique for proving a theorem, whether it is a new or a long established result. Every interesting theorem has in its proof one or more elements of novelty. In fact, it has been remarked that the only interesting things in mathematics are new proofs of old theorems and old proofs of new theorems. Like all aphorisms, this one should be taken _cum grano salis_. But mathematics is very close to music in this respect, at least to classical music. One listens for the familiar and enjoys it when one hears it. Dissonances are invented for the purpose of resolving them. Problems may be invented simply for the pleasure of solving them.

Anyway, mathematicians have more fun than anybody.

1. Quarternions

(a) $q = \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}$, $z = x + iy$, $w = w + iv$;

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$$\begin{aligned}\delta(q) &= z\bar{z} + \bar{w}w = (x + iy)(x - iy) + (u - iv)(u + iv)\\ &= x^2 - i^2y^2 + u^2 - i^2v^2\\ &= x^2 + y^2 + u^2 + v^2.\end{aligned}$$

Since $x, y, u,$ and v are real numbers, $x^2, y^2, u^2,$ and v^2 are nonnegative real numbers. Thus, if $\delta(q) = 0$ then the sum of the nonnegative numbers $x^2, y^2, u^2,$ and v^2 is 0, and so each of these numbers must be zero. Thus, $x, y, u,$ and v must each be zero, so $z, w, \bar{z},$ and $\bar{w}$ are all zero. Therefore, $q = \underline{0}$. Conversely, if $q = \underline{0}$, then $z, w, \bar{z},$ and $\bar{w}$ all are zero, and this implies that $x, y, u,$ and v are all zero. Thus, $x^2, y^2, u^2,$ and v^2 are all zero, so $\delta(q) = 0$.

(b) If $q \neq \underline{0}$ then $\delta(q) \neq 0$ by (a). Now q has an inverse if and only if $\delta(q) \neq 0$. Thus, q has an inverse. Conversely, if q has an inverse, then $\delta(q) \neq 0$, and so by (a), $q \neq \underline{0}$. The form of q^{-1} is

$$q^{-1} = \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}^{-1} = \frac{1}{\delta(q)} \begin{bmatrix} \bar{z} & -w \\ \bar{w} & z \end{bmatrix} = \frac{1}{z\bar{z} + \bar{w}w} \begin{bmatrix} \bar{z} & -w \\ \bar{w} & z \end{bmatrix}.$$

(c) $$\begin{aligned} xI + yU + uV + vW &= x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + u\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + v\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\\ &= \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} + \begin{bmatrix} iy & 0 \\ 0 & -iy \end{bmatrix} + \begin{bmatrix} 0 & u \\ -u & 0 \end{bmatrix} + \begin{bmatrix} 0 & iv \\ iv & 0 \end{bmatrix}\\ &= \begin{bmatrix} x + iy + 0 + 0 & 0 + 0 + u + iv \\ 0 + 0 - u + iv & x - iy + 0 + 0 \end{bmatrix} = \begin{bmatrix} x + iy & u + iv \\ -u + iv & x - iy \end{bmatrix}\\ &= \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} = q. \end{aligned}$$

(d) $$U^2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I,$$

$$V^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I,$$

$$W^2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I;$$

$$UV = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = W,$$

$$-VU = -1\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\begin{bmatrix}i & 0\\0 & -i\end{bmatrix} = -1\begin{bmatrix}0 & -i\\-i & 0\end{bmatrix} = \begin{bmatrix}0 & i\\i & 0\end{bmatrix} = W;$$

$$VW = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\begin{bmatrix}0 & i\\i & 0\end{bmatrix} = \begin{bmatrix}i & 0\\0 & -i\end{bmatrix} = U,$$

$$-WV = -1\begin{bmatrix}0 & i\\i & 0\end{bmatrix}\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} = -1\begin{bmatrix}-i & 0\\0 & i\end{bmatrix} = \begin{bmatrix}i & 0\\0 & -i\end{bmatrix} = U;$$

$$WU = \begin{bmatrix}0 & i\\i & 0\end{bmatrix}\begin{bmatrix}i & 0\\0 & -i\end{bmatrix} = \begin{bmatrix}0 & -i^2\\i^2 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} = V,$$

$$-UW = -1\begin{bmatrix}i & 0\\0 & -i\end{bmatrix}\begin{bmatrix}0 & i\\i & 0\end{bmatrix} = -1\begin{bmatrix}0 & i^2\\-i^2 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} = V.$$

(e) Let $q = \begin{bmatrix}z & w\\-\bar{w} & \bar{z}\end{bmatrix}$, $z = x + iy$, $w = u + iv$;

$r = \begin{bmatrix}m & n\\-\bar{n} & \bar{m}\end{bmatrix}$, $m = a + ib$, $n = c + id$.

From (c), we have

$$q = xI + yU + uV + vW = \begin{bmatrix}x + iy & u + iv\\-(u - iv) & x - iy\end{bmatrix},$$

$$r = aI + bU + cV + dW = \begin{bmatrix}a + ib & c + id\\-(c - id) & a - ib\end{bmatrix}.$$

Therefore,

$$\begin{aligned}q + r &= xI + aI + yU + bU + uV + cV + vW + dW\\ &= (x + a)I + (y + b)U + (u + c)V + (v + d)W\\ &= \begin{bmatrix}(x + a) + i(y + b) & (u + c) + i(v + d)\\-[(u + c) - i(v + d)] & (x + a) - i(y + b)\end{bmatrix}.\end{aligned}$$

Letting

$$s = (x + a) + i(y + b) = z + m,$$

$$t = (u + c) + i(v + d) = w + n,$$

we have

$$q + r = \begin{bmatrix}s & t\\-\bar{t} & \bar{s}\end{bmatrix},$$

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which is in the form of a quaternion, since $s \in C$ and $t \in C$.

Similarly,

$$q - r = \begin{bmatrix} (x-a) + i(y-b) & (u-c) + i(v-d) \\ -[(u-c) - i(v-d)] & (x-a) - i(y-b) \end{bmatrix}.$$

Letting $s' = (x-a) + i(y-b) = z - m$ and $t' = (u-c) + i(v-d) = w - n$, $s' \in C$, $t' \in C$, we have

$$q - r = \begin{bmatrix} s' & t' \\ -\bar{t}' & \bar{s}' \end{bmatrix},$$

which is in the form of a quarternion.

Using the same notation, we have

$$\begin{aligned} qr &= (xI + yU + uV + vW)(aI + bU + cV + dW) \\ &= xaI^2 + xbIU + xcIV + xdIW \\ &\qquad + yaUI + ybU^2 + ycUV + ydUW \\ &\qquad + uaVI + ubVU + ucV^2 + udVW \\ &\qquad + vaWI + vbWU + vcWV + vdW^2 \\ &= xaI + xbU + xcV + xdW + yaU + ybU^2 + ycUV + ydUW \\ &\qquad + uaV + ubVU + ucV^2 + udVW + vaW + vbWU + vcWV + vdW^2 \\ &= xaI + xbU + xcV + xdW + yaU + yb(-I) + ycW + yd(-V) \\ &\qquad + uaV + ub(-W) + uc(-I) + udU + vaW + vbV + vc(-U) + vd(-I) \\ &= (xa - yb - uc - vd)I + (xb + ya + ud - vc)U \\ &\qquad + (xc - yd + ua + vb)V + (xd + yc - ub + va)W \\ &= \begin{bmatrix} A & B \\ P & D \end{bmatrix}, \end{aligned}$$

where

$$A = (xa - yb - uc - vd) + i(xb + ya + ud - vc),$$

$$B = (xc - yd + ua + vb) + i(xd + yc - ub + va),$$

$$P = -\left[(xc - yd + ua + vb) - i(xd + yc - ub + va)\right],$$

$$D = (xa - yb - uc - vd) - i(xb + ya + ud - vc) \quad .$$

Since

$$A = zm - wn, \qquad B = zn + \overline{w}\overline{m}, \quad A \in C, \; B \in C,$$
$$P = -\overline{B}, \qquad D = \overline{A},$$

we have

$$qr = \begin{bmatrix} A & B \\ -\overline{B} & \overline{A} \end{bmatrix},$$

which is in the form of a quarternion.

(f) $\dfrac{1}{|q|^2}\,\overline{q} = \left(\dfrac{1}{[\delta(q)]^{1/2}}\right)^2 \overline{q} = \dfrac{1}{\delta(q)}\,\overline{q}.$

Since

$$q = \begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix},$$

by definition

$$\delta(q) = z\overline{z} + w\overline{w},$$

and therefore

$$\frac{1}{\delta(q)}\,\overline{q} = \frac{1}{z\overline{z} + \overline{w}w} \begin{bmatrix} \overline{z} & -w \\ \overline{w} & z \end{bmatrix},$$

which by definition is q^{-1}.

Now,

$$q^{-1} = \frac{1}{z\overline{z} + \overline{w}w} \begin{bmatrix} \overline{z} & -w \\ \overline{w} & z \end{bmatrix} = \frac{1}{x^2 + y^2 + u^2 + v^2} \begin{bmatrix} x - iy & -u - iv \\ u - iv & x + iy \end{bmatrix}$$

$$= \begin{bmatrix} A & B \\ P & D \end{bmatrix},$$

where

$$A = \frac{x}{x^2 + y^2 + u^2 + v^2} - i\,\frac{y}{x^2 + y^2 + u^2 + v^2}$$

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$$B = -\left(\frac{u}{x^2 + y^2 + u^2 + v^2} + i\,\frac{v}{x^2 + y^2 + u^2 + v^2}\right)$$

$$P = \frac{u}{x^2 + y^2 + u^2 + v^2} - i\,\frac{v}{x^2 + y^2 + u^2 + v^2}$$

$$D = \frac{x}{x^2 + y^2 + u^2 + v^2} + i\,\frac{y}{x^2 + y^2 + u^2 + v^2} .$$

Letting

$$s = \frac{x}{x^2 + y^2 + u^2 + v^2} + i\,\frac{y}{x^2 + y^2 + u^2 + v^2}$$

and

$$t = \frac{u}{x^2 + y^2 + u^2 + v^2} + i\,\frac{v}{x^2 + y^2 + u^2 + v^2} ,$$

we have $s \in C$, $t \in C$, so

$$q^{-1} = \begin{bmatrix} \overline{s} & -t \\ \overline{t} & s \end{bmatrix}$$

is in the form of a quarternion. Since $A = \overline{s}$ and $B = -t$, we get

$$q^{-1} = \begin{bmatrix} A & B \\ -\overline{B} & \overline{A} \end{bmatrix} ,$$

which is perhaps a clearer way of expressing q^{-1} in quarternion form.

(g) $$q^2 - t(q)q + |q|^2\, I = \begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix}^2 - (2x) \begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix} + \left(\left[\delta(q)\right]^{1/2}\right)^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} z^2 - w\overline{w} & zw + w\overline{z} \\ -\overline{w}z - \overline{z}\overline{w} & -\overline{w}w + \overline{z}^2 \end{bmatrix} + \begin{bmatrix} -2xz & -2xw \\ 2x\overline{w} & -2x\overline{z} \end{bmatrix} + (x^2 + y^2 + v^2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} A & B \\ P & D \end{bmatrix} + \begin{bmatrix} -2x^2 - 2xiy & -2xu - 2xiv \\ 2xu - 2xiv & -2x^2 + 2xiy \end{bmatrix}$$

$$+ \begin{bmatrix} x^2 + y^2 + u^2 + v^2 & 0 \\ 0 & x^2 + y^2 + w^2 + v^2 \end{bmatrix}$$

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where

$$A = x^2 + 2xiy - y^2 - u^2 + uiv - ivu - v^2$$

$$B = xu + xiv + iyu - yv + ux - uiy + ivx + vy$$

$$P = - ux - uiy + ivx + vy - xu + xiv + iyu - vy$$

$$D = - u^2 - uiv + ivu - v^2 + x^2 - 2xiy - y^2.$$

Hence we obtain

$$q^2 - t(q)q + |q|^2 I = \begin{bmatrix} A' & B' \\ P' & D' \end{bmatrix}$$

where

$$A' = x^2 + 2xiy - y^2 - u^2 - v^2 - 2x^2 - 2xiy + x^2 + y^2 + u^2 + v^2 = 0$$

$$B' = 2xu + 2xiv - 2xu - 2xiv + 0 = 0$$

$$P' = - 2xu + 2xiv + 2xu - 2xiv + 0 = 0$$

$$D' = - u^2 - v^2 + x^2 - 2xiy - y^2 - 2x^2 + 2xiy + x^2 + y^2 + u^2 + v^2 = 0$$

and therefore

$$q^2 - t(q)q + |q|^2 I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \underline{0}.$$

(h) $q\bar{q} = (aI + bU + cV + dW)(aI - bU - cV - dW)$

$$= a^2I^2 - abIU - acIV - adIW + baUI - b^2U^2 - bcUV - bdUW$$
$$+ caVI - cbVU - c^2V^2 - cdVW + daWI - dbWU - dcWV - d^2W^2.$$

Using (d), then, we get

$$q\bar{q} = a^2I - abU - acV - adW + baU - b^2(-I) - bcW - bd(-V)$$
$$+ caV - cb(-W) - c^2(-I) - cdU + daW - dbV - dc(-U) - d^2(-I)$$
$$= a^2I + b^2I + c^2I + d^2I = (a^2 + b^2 + c^2 + d^2)I$$
$$= \delta(q)I$$
$$= \left([\delta(q)]^{1/2}\right)^2 I = |q|^2 I.$$

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(i) Let $q = xI + yU + uV + vW$ and $r = aI + bU + cV + dW$. Since $q \in Q$ and $r \in Q$ it follows by (e) that $qr \in Q$. Referring back to part (e), we see that

$$qr = (xa - yb - uc - vd)I + (xb + ya + ud - vc)U$$
$$+ (xc - yd + ua + vb)V + (xd + yc - ub + va)W.$$

Using (a), we obtain

$$\delta(q) = x^2 + y^2 + u^2 + v^2 \quad \text{and} \quad \delta(r) = a^2 + b^2 + c^2 + d^2.$$

Also,

$$\delta(qr) = (xa - yb - uc - vd)^2 + (xb + ya + ud - vc)^2$$
$$+ (xc - yd + ua + vb)^2 + (xd + yc - ub + va)^2.$$

Thus,

$$|q| = \left[\delta(q)\right]^{1/2} = \left[x^2 + y^2 + u^2 + v^2\right]^{1/2} \quad \text{and}$$
$$|r| = \left[\delta(r)\right]^{1/2} = \left[a^2 + b^2 + c^2 + d^2\right]^{1/2}.$$

Therefore,

$$|q|\,|r| = \left[x^2 + y^2 + u^2 + v^2\right]^{1/2} \left[a^2 + b^2 + c^2 + d^2\right]^{1/2}$$
$$= \left[(x^2 + y^2 + u^2 + v^2)(a^2 + b^2 + c^2 + d^2)\right]^{1/2}$$
$$= \left[x^2(a^2 + b^2 + c^2 + d^2) + y^2(a^2 + b^2 + c^2 + d^2)\right.$$
$$\left. + u^2(a^2 + b^2 + c^2 + d^2) + v^2(a^2 + b^2 + c^2 + d^2)\right]^{1/2}.$$

Now,

$$|qr| = \left[\delta(qr)\right]^{1/2} = \left[(xa - yb - uc - vd)^2 + (xb + ya + ud - vc)^2\right.$$
$$\left. + (xc - yd + ua + vb)^2 + (xd + yc - ub + va)^2\right]^{1/2}$$
$$= \left[x^2a^2 - 2xyab - 2xuac - 2xvad + y^2b^2 + 2yubc + 2yvbd + u^2c^2\right.$$
$$+ 2uvcd + v^2d^2 + x^2b^2 + 2xyab + 2xubd - 2xvbc + y^2a^2$$

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$$+ 2yuad - 2yvac + u^2d^2 - 2uvcd + v^2c^2 + x^2c^2 - 2xycd$$
$$+ 2xuac + 2xvbc + y^2d^2 - 2yuad - 2yvbd + u^2a^2 + 2uvab + v^2b^2$$
$$+ x^2d^2 + 2xycd - 2xubd + 2xvad + y^2c^2 - 2yubc + 2yvac + u^2b^2$$
$$- 2uvab + v^2a^2\Big]^{1/2}$$
$$= \Big[x^2a^2 + x^2b^2 + x^2c^2 + x^2d^2 + y^2b^2 + y^2a^2 + y^2d^2 + y^2c^2 + u^2c^2$$
$$+ u^2d^2 + u^2a^2 + u^2b^2 + v^2d^2 + v^2c^2 + v^2b^2 + v^2a^2\Big]^{1/2}$$
$$= \Big[x^2(a^2 + b^2 + c^2 + d^2) + y^2(a^2 + b^2 + c^2 + d^2)$$
$$+ u^2(a^2 + b^2 + c^2 + d^2) + v^2(a^2 + b^2 + c^2 + d^2)\Big]^{1/2}$$
$$= |q|\ |r|.$$

Since $q \in Q$ and $r \in Q$ it follows from (e) that $q + r \in Q$. Now

$$q + r = (x + a)I + (y + b)U + (u + c)V + (v + d)W,$$
$$\delta(q + r) = (x + a)^2 + (y + b)^2 + (u + c)^2 + (v + d)^2.$$

The inequality

$$|q + r| \leq |q| + |r|$$

is equivalent to

$$\left[(x + a)^2 + (y + b)^2 + (w + c)^2 + (v + d)^2\right]^{1/2}$$
$$\leq \left[x^2 + y^2 + u^2 + v^2\right]^{1/2} + \left[a^2 + b^2 + c^2 + d^2\right]^{1/2},$$

or to

$$x^2 + 2ax + a^2 + y^2 + 2by + b^2 + u^2 + 2cu + c^2 + v^2 + 2dv + d^2$$
$$\leq x^2 + y^2 + u^2 + v^2 + 2\left[x^2 + y^2 + u^2 + v^2\right]^{1/2}\left[a^2 + b^2 + c^2 + d^2\right]^{1/2}$$
$$+ a^2 + b^2 + c^2 + d^2,$$

or to

$$ax + by + cu + dv \leq \left[x^2 + y^2 + u^2 + v^2\right]^{1/2}\left[a^2 + b^2 + c^2 + d^2\right]^{1/2}.$$

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This is valid if the left-hand member is negative. Otherwise, it is equivalent to

$$a^2x^2 + b^2y^2 + c^2u^2 + d^2v^2 + 2abxy + 2acxu + 2adxv + 2bcyu + 2bdyv + 2cduv$$
$$\leq a^2x^2 + b^2y^2 + c^2u^2 + d^2v^2 + a^2y^2 + a^2u^2 + a^2v^2 + b^2x^2 + b^2u^2$$
$$+ b^2v^2 + c^2x^2 + c^2y^2 + c^2v^2 + d^2x^2 + d^2y^2 + d^2u^2,$$

or to

$$0 \leq (ay - bx)^2 + (au - cx)^2 + (av - dx)^2 + (bu - cy)^2 + (bv - dy)^2 + (cv - du)^2,$$

which is valid since the right-hand side is a sum of squares of real numbers.

You should compare this result with the "triangle inequality" (2) on page 163 of the text, and should note that the present proof merely generalizes the proof on pages 163 and 164 of the text.

2. Nonassociative Algebras

(a) (i) $A \circ B = AB - BA,$

$$-B \circ A = -(BA - AB) = -BA + AB = AB - BA = A \circ B.$$

(ii) $A \circ A = AA - AA = \underline{0}.$

(iii) $A \circ (B \circ C) + B \circ (C \circ A) + C \circ (A \circ B)$

$$= A \circ (BC - CB) + B \circ (CA - AC) + C \circ (AB - BA)$$
$$= A(BC - CB) - (BC - CB)A + B(CA - AC) - (CA - AC)B + C(AB - BA) - (AB - BA)C$$
$$= ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB + CAB - CBA - ABC + BAC$$
$$= \underline{0}.$$

(iv) $A \circ I = AI - IA = A - A = \underline{0},$

$$I \circ A = IA - AI = A - A = \underline{0}.$$

(b) Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Then

$$A \circ (B \circ C) = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \circ \left(\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \circ \left(\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \circ \left(\begin{bmatrix} 0 & 1 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \circ \begin{bmatrix} -3 & 0 \\ -6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ -6 & 3 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -15 & 6 \\ -9 & 0 \end{bmatrix} - \begin{bmatrix} -3 & -6 \\ 3 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 12 \\ -12 & 12 \end{bmatrix},$$

and

$$(A \circ B) \circ C = \left(\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \right) \circ \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \right) \circ \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 7 & 2 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 6 & 6 \end{bmatrix} \right) \circ \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ -3 & -6 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 \\ -3 & -6 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ -3 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ -6 & -15 \end{bmatrix} - \begin{bmatrix} -3 & -6 \\ 0 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 12 \\ -6 & -3 \end{bmatrix}.$$

Hence, $A \circ (B \circ C) \neq (A \circ B) \circ C$ since $\begin{bmatrix} -12 & 12 \\ -12 & 12 \end{bmatrix} \neq \begin{bmatrix} 3 & 12 \\ -6 & -3 \end{bmatrix}$.

(c) $A \circ (B + C) = A(B + C) - (B + C)A$

$= AB + AC - BA - CA$

$= AB - BA + AC - CA$

$= (A \circ B) + (A \circ C).$

$(A + B) \circ C = (A + B)C - C(A + B)$

$= AC + BC - CA - CB$

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$$= AC - CA + BC - CB$$

$$= (A \circ C) + (B \circ C).$$

(d) Let x be a number. Then

$$x^2(A \circ B) = x^2(AB - BA) = x^2AB - x^2BA$$

$$= xAxB - xBxA$$

$$= xA \circ xB.$$

(e) Suppose there were an $\circ$ unit. Call it I'. Then, for any A, we would have

$$A \circ I' = A = I' \circ A,$$

by definition of unit. If $A = \underline{0}$, then

$$I' \circ \underline{0} = I'\underline{0} - \underline{0}I' = \underline{0} - \underline{0} = \underline{0},$$

and

$$\underline{0} \circ I' = \underline{0}I' - I'\underline{0} = \underline{0} - \underline{0} = \underline{0}.$$

Certainly $I' \neq \underline{0}$. Suppose $A \neq \underline{0}$.
We know that

$$A \circ I' = I' \circ A,$$

by definition. Therefore,

$$AI' - I'A = I'A - AI',$$

so

$$2AI' = 2I'A$$

and

$$2(AI' - I'A) = \underline{0},$$

which implies that

$$AI' - I'A = \underline{0},$$

and thus that

$$A \circ I' = \underline{0}.$$

But

$$A \circ I' = A.$$

Together these imply that $A = \underline{0}$, which by hypothesis is untrue. Therefore, there is no $\circ$ unit.

(f) $AjB = \frac{(AB + BA)}{2}$.

(i) $AjA = \frac{AA + AA}{2} = \frac{2AA}{2} = A^2$.

(ii) $AjI = \frac{AI + IA}{2} = \frac{A + A}{2} = A = IjA$,

so I is a j unit.

(iii) $Aj(B + C) = \frac{A(B + C) + (B + C)A}{2} = \frac{AB + AC + BA + CA}{2}$

$= \frac{AB + BA}{2} + \frac{AC + CA}{2}$

$= AjB + AjC$.

(iv) $(A + B)jC = \frac{(A + B)C + C(A + B)}{2} = \frac{AC + BC + CA + CB}{2}$

$= \frac{AC + CA}{2} + \frac{BC + CB}{2}$

$= AjC + BjC$.

(v) For any number x,

$$x^2(AjB) = x^2\left(\frac{AB + BA}{2}\right)$$

$$= \frac{x^2AB + x^2BA}{2} = \frac{xAxB + xBxA}{2}$$

$$= xA \ j \ xB.$$

(vi.) $Aj(BC) - (AB)jC + Bj(CA) - (BC)jA + Cj(AB) - (CA)jB$

$$= \frac{1}{2}\Big[ABC + BCA - ABC - CAB + BCA + CAB - BCA - ABC + CAB + ABC - CAB - BCA\Big]$$

$$= \frac{1}{2}\left[\underline{0}\right] = \underline{0}.$$

3. The Algebra of Subsets

(a) (i) $\{\underline{0}\} + \{\underline{0}\} = \{\underline{0}\}$, so $\{\underline{0}\}$ is an additive subset.

(ii) $\{I\} + \{I\} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \not\subset \{I\}$, so $\{I\}$ is not an additive subset.

(iii) $M + M \subset M$, since the sum of any two 2×2 matrices is always a 2×2 matrix, so M is an additive subset.

(iv) $Z + Z \subset Z$, since the sum of any two complex numbers is a complex number and Z is the set of 2×2 matrices which is isomorphic with the set of complex numbers, so Z is an additive subset.

(v) $M_1 + M_1 \not\subset M_1$.

To see this, consider

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad \delta(A) = 2 - 1 = 1,$$

and

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \delta(B) = 2 - 1 = 1.$$

Thus $A \in M_1$, $B \in M_1$. Now

$$A + B \in M_1 + M_1,$$

and

$$A + B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix},$$

$$\delta(A + B) = (3)(3) - (2)(2) = 9 - 4 = 5 \neq 1.$$

Accordingly, by counterexample, M_1 is not an additive subset.

(vi) Let $\underline{P}$ be the set of all elements of M with nonnegative entries. Then

$\underline{P} + \underline{P} \subset \underline{P}$ since the sum of nonnegative numbers is a nonnegative number. Thus $\underline{P}$ is an additive subset.

(b) (i) $\underline{A} + \underline{B} = \{A + B \mid A \in \underline{A} \text{ and } B \in \underline{B}\} = \{B + A \mid B \in \underline{B} \text{ and } A \in \underline{A}\} = \underline{B} + \underline{A}$, since ordinary matrix addition is commutative.

(ii) $\underline{A} + (\underline{B} + \underline{C}) = \{A + (B + C) : A \in \underline{A},\ B \in \underline{B}, \text{ and } C \in \underline{C}\}$

$= \{(A + B) + C : A \in \underline{A},\ B \in \underline{B}, \text{ and } C \in \underline{C}\}$,

since ordinary matrix addition is associative.

(iii) $\underline{A} + \underline{C} = \{A + C : A \in \underline{A} \text{ and } C \in \underline{C}\}$

$\subset \{B + C : B \in \underline{B} \text{ and } C \in \underline{C}\}$ since $\underline{A} \subset \underline{B}$

$= \underline{B} + \underline{C}$.

(c) $(\underline{A} + \underline{B}) + (\underline{A} + \underline{B}) = \{(A + B) + (A + B) : A + B \in \underline{A} + \underline{B} \text{ and } A + B \in \underline{A} + \underline{B}\}$

$= \{A + (B + A) + B : A \in \underline{A},\ B \in \underline{B}\}$

$= \{A + (A + B) + B : A \in \underline{A},\ B \in \underline{B}\}$

$= \{(A + A) + (B + B) : A \in \underline{A},\ B \in \underline{B}\}$

$\subset \{A + B : A \in \underline{A},\ B \in \underline{B}\} = \underline{A} + \underline{B}$,

since A and B are both additive subsets.

(d) Let

$$\underline{U} = \left\{ A : A \in M \text{ and } Av = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\},$$

where v is fixed. Then

$$\begin{aligned}
\underline{U} + \underline{U} &= \{A + B : A \in \underline{U} \text{ and } B \in \underline{U}\} \\
&= \left\{ A + B : A \in M,\ B \in M,\ Av = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ Bv = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \\
&= \left\{ A + B : (A + B) \in M,\ (A + B)v = Av + Bv = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \\
&\subset \underline{U},
\end{aligned}$$

so $\underline{U}$ is an additive subset of M.

Let $Av = 0$. Then $(-A)v = (-1)Av = (-1)\underline{0} = \underline{0}$.

(e) (i) $\{\underline{0}\}\{\underline{0}\} = \{\underline{0}\}$, so $\{\underline{0}\}$ is a multiplicative subset.

(ii) $\{I\}\{I\} = \{I\}$, so $\{I\}$ is a multiplicative subset.

(iii) $M \cdot M \subset M$, since the product of any two 2×2 matrices is always a 2×2 matrix; so M is a multiplicative subset.

(iv) $ZZ \subset Z$, since the product of any two complex numbers is a complex number and Z is the set of 2×2 matrices that is isomorphic with the set of complex numbers; so Z is a multiplicative subset.

(v) $M_1M_1 \subset M_1$. Let $A \in M_1$, $B \in M_1$ so $\delta(A) = \delta(B) = 1$. Then $\delta(AB) = \delta(A)\delta(B) = 1 \bullet 1 = 1$, so M_1 is a multiplicative subset.

(vi) $\underline{P}\underline{P} \subset \underline{P}$, since the product and the sum of nonnegative numbers are nonnegative numbers; so $\underline{P}$ is a multiplicative subset.

(f) (i)
$$\begin{aligned}
\underline{A}(\underline{BC}) &= \{A(BC) : A \in \underline{A},\ B \in \underline{B}, \text{ and } C \in \underline{C}\} \\
&= \{(AB)C : A \in \underline{A},\ B \in \underline{B}, \text{ and } C \in \underline{C}\} \\
&= (\underline{AB})\underline{C},
\end{aligned}$$

since ordinary matrix multiplication is associative.

(ii)
$$\begin{aligned}
\underline{AC} &= \{AC : A \in \underline{A} \text{ and } C \in \underline{C}\} \\
&\subset \{BC : B \in \underline{B} \text{ and } C \in \underline{C}\} \quad (\text{since } \underline{A} \subset \underline{B}) \\
&= \underline{BC}.
\end{aligned}$$

(g) Let $\underline{A} = \left\{A : A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \text{ where } a \in R \text{ and } a \neq 0\right\}$.

Let $\underline{B} = \left\{B : B = \begin{bmatrix} b & 1 \\ 1 & 1 \end{bmatrix}, \text{ where } b \in R\right\}$.

Then $\underline{AB} = \{AB : A \in \underline{A}, B \in \underline{B}\}$; $\underline{BA} = \{BA \mid B \in \underline{B}, A \in \underline{A}\}$;

$$AB = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\begin{bmatrix} b & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}; \quad BA = \begin{bmatrix} 0 & ba \\ 0 & a \end{bmatrix}.$$

Thus $AB = BA$ means $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ba \\ 0 & a \end{bmatrix}$, and this means $a = 0$ and $a = ba$.

But $a \neq 0$ by hypothesis, so $AB \neq BA$.

Therefore $\underline{AB} \neq \underline{BA}$.

(h) (i) Let $\underline{A} = \{\underline{0}, I\}$. Then

$$\underline{AA} = \{\underline{0}\underline{0}, \underline{0}I, I\underline{0}, II\} = \{\underline{0}, \underline{0}, \underline{0}, I\} = \{\underline{0}, I\} = \underline{A},$$

so $\underline{A}$ is a multiplicative subset.

(ii) Let $\underline{B} = \{I, -I\}$. Then

$$\underline{BB} = \{II, I(-I), (-I)I, (-I)(-I)\} = \{I, -I, -I, I\} = \{I, -I\} = \underline{B},$$

so $\underline{B}$ is a multiplicative subset.

(iii) Let $\underline{N}$ be the set of all elements of M with negative entries. Then $\underline{NN} \not\subset \underline{N}$, since the product of two negative numbers is a positive number; so $\underline{N}$ is not a multiplicative subset.

(iv) Let $\underline{E}$ be the set of all elements of M for which the upper left-hand entry is less than 1.

Let

$$C = \begin{bmatrix} -2 & 0 \\ 1 & 2 \end{bmatrix} \text{ and let } D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

so that $C \in \underline{E}$, $D \in \underline{E}$. Then

$$CD = \begin{bmatrix} -2 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} \not\in \underline{E},$$

since $2 \not< 1$.

Therefore $\underline{E}$ is not a multiplicative subset.

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However, consider the subset $\underline{F}$ of $\underline{E}$ for which the upper left-hand entry is between 0 and 1, inclusive, and the lower left-hand entry is 0. Then $\underline{F} = \{F : F = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, where $a, b, c \in R$ and $0 \leq a \leq 1\}$. Consider

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix},$$

where $A \in \underline{F}$, $B \in \underline{F}$. Then

$$AB = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} ax & ay + bz \\ 0 & cz \end{bmatrix} = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix},$$

where

$$d, e, f \in R \quad \text{and} \quad 0 \leq d = ax \leq 1,$$

so that $AB \in \underline{F}$. Thus, $\underline{F}$ is a multiplicative subset.

(v) Let

$$\underline{G} = \left\{G : G \in M \text{ and } G = \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}, \text{ where } 0 \leq x,\ 0 \leq y \text{ and } x + y \leq 1\right\}.$$

Consider

$$H = \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} w & z \\ 0 & 1 \end{bmatrix},$$

where $H \in \underline{G}$, $J \in \underline{G}$. Thus, $0 \leq x$, $0 \leq y$, $x + y \leq 1$, $0 \leq w$, $0 \leq z$, and $w + z \leq 1$. Now,

$$HJ = \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}\begin{bmatrix} w & z \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} xw & xz + y \\ 0 & 1 \end{bmatrix}.$$

Since $0 \leq x$ and $0 \leq w$, we have $0 \leq xw$.

Also, $0 \leq x$, $0 \leq z$ implies $0 \leq xz$ and $0 \leq y$ implies $0 \leq xz + y$. Then $xw + xz + y = x(w + z) + y \leq x(1) + y = x + y \leq 1$. Therefore,

$$HJ = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix},$$

where $0 \leq a$, $0 \leq b$, and $a + b \leq 1$, so $HJ \subset \underline{G}$. Thus $\underline{G}$ $\underline{\text{is}}$ a multiplicative subset.

[pages 226, 227]

Additional Results:

Let $\underline{tA} = \{tA \mid t \in R$ and t is fixed, $A \in \underline{A} \in M\}$. Then $\underline{tA}$ is obviously an additive subset if $\underline{A}$ is additive

$$tB + tC = t(B + C) = tD \in \underline{tA} \quad \text{for } B, C, \text{ and } D \in \underline{A}$$

(B and C are taken in $\underline{A}$; and, since $\underline{A}$ is additive, $B + C = D \in \underline{A}$). Since $tBtC = t^2BC \neq tBC$ unless $t^2 = t$, $\underline{tA}$ is not multiplicative unless $t^2 = t$ <u>and</u> $\underline{A}$ is multiplicative.

Define $(-1)\underline{A} = \{(-1)A : A \in \underline{A}\}$.

Define $-\underline{A} = \{-A : A \in \underline{A}\}$. [$-A$ is a well-defined matrix.]

Then $(-1)\underline{A} = -\underline{A}$, since $-A = (-1)A$.

Define $\underline{A}^n = \{A^n : A \in \underline{A}\}$. In general, $\underline{A}^n$ is neither additive nor multiplicative.

$$\begin{aligned}\underline{A}(\underline{B} + \underline{C}) &= \{A(B + C) : A \in \underline{A}, B \in \underline{B}, C \in \underline{C}\}\\ &= \{AB + AC : A \in \underline{A}, B \in \underline{B}, C \in \underline{C}\}\\ &= \underline{AB} + \underline{AC}, \text{ since matrix multiplication distributes.}\end{aligned}$$

Consider $\underline{A}(\underline{B} \cup \underline{C})$. $\cup$ = set union, $\cap$ = set intersection.

$$M \in \underline{B} \cup \underline{C} \longrightarrow M \in \underline{B} \text{ or } M \in \underline{C} \text{ (or } M \in (\underline{B} \cap \underline{C})).$$

$$\begin{aligned}M \in \underline{A}(\underline{B} \cup \underline{C}) &\longleftrightarrow M \in \underline{AB} \text{ or } M \in \underline{AC} \text{ (or } M \in \underline{A}(\underline{B} \cap \underline{C}))\\ &\longleftrightarrow M \in \underline{AB} \cup \underline{AC}.\end{aligned}$$

Therefore, $\underline{A}(\underline{B} \cup \underline{C}) = \underline{AB} \cup \underline{AC}$.

$$\begin{aligned}M \in \underline{A}(\underline{B} \cap \underline{C}) &\longleftrightarrow M \in \underline{AB} \text{ and } M \in \underline{AC}\\ &\longleftrightarrow M \in \underline{AB} \cap \underline{AC}.\end{aligned}$$

Therefore, $\underline{A}(\underline{B} \cap \underline{C}) = \underline{AB} \cap \underline{AC}$.

4. Analysis and Synthesis of Proofs

(a) (i) $\left.\begin{aligned}x \wedge y &= \text{the smaller of } x \text{ and } y\\ y \wedge x &= \text{the smaller of } y \text{ and } x\end{aligned}\right\}$ Clearly, these two are the same.

[pages 226-228]

(ii) $x \vee y$ = the larger of x and y } Clearly, these two are
$y \vee x$ = the larger of y and x } the same.

(iii) $x \wedge (y \wedge z)$ = the smaller of x and the smaller of y and z,

$(x \wedge y) \wedge z$ = the smaller of the smaller of x and y and z; clearly, each of these is the smallest of the three numbers x, y, and z; so the two expressions are the same.

(iv) $x \vee (y \vee z) = (x \vee y) \vee z$. Each of these expressions calls for the largest of the three numbers x, y, and z, so they are the same.

(v) $x \wedge x = x$. The smaller of x and x is certainly x.

(vi) $x \vee x = x$. Similarly, the larger of x and x must be x.

(vii) $x \wedge (y \vee z)$ = the smaller of x and the larger of y and z,
$(x \wedge y) \vee (x \wedge z)$ = the larger of the smaller of x and y and the smaller of x and z.

If further proof is desired:

Case I	$x < y < z$	$x \wedge (y \vee z) = x \wedge z = x$	$(x \wedge y) \vee (x \wedge z) = x \vee x = x$
" II	$x < z < y$	" $= x \wedge y = x$	" $= x \vee x = x$
" III	$y < x < z$	" $= x \wedge z = x$	" $= y \vee x = x$
" IV	$y < z < x$	" $= x \wedge z = z$	" $= y \vee z = z$
" V	$z < x < y$	" $= x \wedge y = x$	" $= x \vee z = x$
" VI	$z < y < x$	" $= x \wedge y = y$	" $= y \vee z = y$

(viii) $x \vee (y \wedge z)$ = the larger of x and the smaller of y and z,

$(x \vee y) \wedge (x \vee z)$ = the smaller of the larger of x and y and the larger of x and z.

Also:

Case I	$x < y < z$	$x \vee (y \wedge z) = x \vee y = y$	$(x \vee y) \wedge (x \vee z) = y \wedge z = y$
" II	$x < z < y$	" $= x \vee z = z$	" $= y \wedge z = z$
" III	$y < x < z$	" $= x \vee y = x$	" $= x \wedge z = x$
" IV	$y < z < x$	" $= x \vee y = x$	" $= x \wedge x = x$
" V	$z < x < y$	" $= x \vee z = x$	" $= y \wedge x = x$
" VI	$z < y < x$	" $= x \vee z = x$	" $= x \wedge x = x$

(b) $$A(BC) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right)$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} xe + yg & xf + yh \\ ze + wg & zf + wh \end{bmatrix}$$

$$= \begin{bmatrix} axe + ayg + bze + bwg & axf + ayh + bzf + bwh \\ cxe + cyg + dze + dwg & cxf + cyh + dzf + dwh \end{bmatrix}$$

$$= \begin{bmatrix} axe + bze + ayg + bwg & axf + bzf + ayh + bwh \\ cxe + dze + cyg + dwg & cxf + dzf + cyh + dwh \end{bmatrix}$$

$$= \begin{bmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$= \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} \right) \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$= (AB)C.$$

$$A \wedge (B \wedge C) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \wedge \left(\begin{bmatrix} x & y \\ z & w \end{bmatrix} \wedge \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right)$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \wedge \begin{bmatrix} (x \wedge e) \vee (y \wedge g) & (x \wedge f) \vee (y \wedge h) \\ (z \wedge e) \vee (w \wedge g) & (z \wedge f) \vee (w \wedge h) \end{bmatrix}$$

$$= \begin{bmatrix} (a \wedge \{(x \wedge e) \vee (y \wedge g)\}) \vee (b \wedge \{(z \wedge e) \vee (w \wedge g)\}) & (a \wedge \{(x \wedge f) \vee (y \wedge h)\}) \vee (b \wedge \{(z \wedge f) \vee (w \wedge h)\}) \\ (c \wedge \{(x \wedge e) \vee (y \wedge g)\}) \vee (d \wedge \{(z \wedge e) \vee (w \wedge g)\}) & (c \wedge \{(x \wedge f) \vee (y \wedge h)\}) \vee (d \wedge \{(z \wedge f) \vee (w \wedge h)\}) \end{bmatrix}$$

$$= \begin{bmatrix} (\{a \wedge (x \wedge e)\} \vee \{a \wedge (y \wedge g)\}) \vee (\{b \wedge (z \wedge e)\} \vee \{b \wedge (w \wedge g)\}) & (\{a \wedge (x \wedge f)\} \vee \{a \wedge (y \wedge h)\}) \vee (\{b \wedge (z \wedge f)\} \vee \{b \wedge (w \wedge h)\}) \\ (\{c \wedge (x \wedge e)\} \vee \{c \wedge \{y \wedge g)\}) \vee (\{d \wedge (z \wedge e)\} \vee \{d \wedge (w \wedge g)\}) & (\{c \wedge (x \wedge f)\} \vee \{c \wedge (y \wedge h)\}) \vee \{(d \wedge (z \wedge f)\} \vee \{d \wedge (w \wedge h)\}) \end{bmatrix}$$

$$= \begin{bmatrix} \{(a \wedge x) \wedge e\} \vee \{(a \wedge y) \wedge g\} \vee \{(b \wedge z) \wedge e\} \vee \{(b \wedge w) \wedge g\} & \{(a \wedge x) \wedge f\} \vee \{(a \wedge y) \wedge h\} \vee \{(b \wedge z) \wedge f\} \vee \{(b \wedge \quad \wedge h\} \\ \{(c \wedge x) \wedge e\} \vee \{(c \wedge y) \wedge g\} \vee \{(d \wedge z) \wedge e\} \vee \{(d \wedge w) \wedge g\} & \{(c \wedge x) \wedge f\} \vee \{(c \wedge y) \wedge h\} \vee \{(d \wedge z) \wedge f\} \vee \{(d \wedge \quad h\} \end{bmatrix}$$

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$$= \begin{bmatrix} \{(a \wedge x) \wedge e\} \vee \{(b \wedge z) \wedge e\} \vee \{(a \wedge y) \wedge g\} \vee \{(b \wedge w) \wedge g\} \\ \{(c \wedge x) \wedge e\} \vee \{(d \wedge z) \wedge e\} \vee \{(c \wedge y) \wedge g\} \vee \{(d \wedge w) \wedge g\} \end{bmatrix.$$

$$\begin{matrix} \{(a \wedge x) \wedge f\} \vee \{(b \wedge z) \wedge f\} \vee \{(a \wedge y) \wedge h\} \vee \{(b \wedge w) \wedge h\} \\ \{(c \wedge x) \wedge f\} \vee \{(d \wedge z) \wedge f\} \vee \{(c \wedge y) \wedge h\} \vee \{(d \wedge w) \wedge h\} \end{matrix}\Bigg]$$

$$= \Bigg[\begin{matrix} (\{(a \wedge x) \vee (b \wedge z)\} \wedge e) \vee (\{(a \wedge y) \vee (b \wedge w)\} \wedge g) \\ (\{(c \wedge x) \vee (d \wedge z)\} \wedge e) \vee (\{(c \wedge y) \vee (d \wedge w)\} \wedge g) \end{matrix}$$

$$\begin{matrix} (\{(a \wedge x) \vee (b \wedge z)\} \wedge f) \vee (\{(a \wedge y) \vee (b \wedge w)\} \wedge h) \\ (\{(c \wedge x) \vee (d \wedge z)\} \wedge f) \vee (\{(c \wedge y) \vee (d \wedge w)\} \wedge h) \end{matrix}\Bigg]$$

$$= \begin{bmatrix} (a \wedge x) \vee (b \wedge z) & (a \wedge y) \vee (b \wedge w) \\ (c \wedge x) \vee (d \wedge z) & (c \wedge y) \vee (d \wedge w) \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$= \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \wedge \begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) \wedge \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$= (A \wedge B) \wedge C.$

(c) $A + B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} a + x & b + y \\ c + z & d + w \end{bmatrix}.$

Define $A \vee B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vee \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} a \vee x & b \vee y \\ c \vee z & d \vee w \end{bmatrix}.$

(d) A few "rules" are as follows (from matrix rules):

(i) $A \vee (B \vee C) = (A \vee B) \vee C$, the analogue of part (b).

$$A \vee (B \vee C) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vee \left(\begin{bmatrix} x & y \\ z & w \end{bmatrix} \vee \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vee \begin{bmatrix} x \vee e & y \vee f \\ z \vee g & w \vee h \end{bmatrix}$$

$$= \begin{bmatrix} a \vee (x \vee e) & b \vee (y \vee f) \\ c \vee (z \vee g) & d \vee (w \vee h) \end{bmatrix}$$

$$= \begin{bmatrix} (a \vee x) \vee e & (b \vee y) \vee f \\ (c \vee z) \vee g & (d \vee w) \vee h \end{bmatrix}$$

$$= \begin{bmatrix} a \vee x & b \vee y \\ c \vee z & d \vee w \end{bmatrix} \vee \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$= \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \vee \begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) \vee \begin{bmatrix} e & f \\ g & h \end{bmatrix} = (A \vee B) \vee C.$$

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(ii) $(A \vee B) = (B \vee A)$.

(iii) $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$.

(iv) $A \vee \underline{0} = A$ <u>only if</u> each entry, a, of A is ≥ 0.

(v) $A \wedge \underline{0} = \underline{0} = \underline{0} \wedge A$ <u>only if</u> each entry of A is ≥ 0.

(vi) $A \wedge I = A = I \wedge A$ <u>only if</u> each entry of A is between 0 and 1 inclusive.

(vii) $A \vee (-A) \neq \underline{0}$.

The following "rules" are also true, though they are not paralleled in matrix algebra.

(i) $A \wedge B = B \wedge A$.

(ii) $A \wedge A = A$.

(iii) $A \vee A = A$.

(iv) $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$.

(v) $A \vee I = I$ <u>only if</u> for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a \leq 1$, $b \leq 0$, $c \leq 0$, $d \leq 1$.